

Separable wave equations for gravitoelectromagnetic perturbations of rotating charged black strings

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Abstract.

Rotating charged black strings are exact solutions of four-dimensional Einstein-Maxwell equations with a negative cosmological constant and a non-trivial spacetime topology. According to the AdS/CFT correspondence, these black strings are dual to rotating thermal states of a strongly interacting quantum field theory with nonzero chemical potential that lives in a cylinder. The dynamics of linear fluctuations in the dual field theory can be studied from the perturbation equations for classical fields in a black-string spacetime. With this motivation in mind, we develop here a completely gauge and tetrad invariant perturbation approach to deal with the gravitoelectromagnetic fluctuations of rotating charged black strings in the presence of sources. As usual, for any charged black hole, a perturbation in the background electromagnetic field induces a metric perturbation and vice versa. In spite of this coupling and the non-vanishing angular momentum, we show that linearization of equations of the Newman-Penrose formalism leads to four separated second-order complex equations for suitable combinations of the spin coefficients, the Weyl and the Maxwell scalars. Then, we generalize the Chandrasekhar transformation theory by the inclusion of sources and apply it to reduce the perturbation problem to four decoupled inhomogeneous wave equations — a pair for each sector of perturbations. The radial part of such wave equations can be put into Schrödinger-like forms after Fourier transforming them with respect to time. We find that the resulting effective potentials form two pairs of supersymmetric partner potentials and, as a consequence, the fundamental variables of one perturbation sector are related to the variables of the other sector. The relevance of such a symmetry in connection to the AdS/CFT correspondence is discussed, and future applications of the perturbation theory developed here are outlined.

1. Introduction

During the last two and a half decades, black holes in asymptotically anti-de Sitter (AdS) spacetimes have been recognized as important objects for the study of the foundations of the gravitational interaction and its connections to other areas of physics. In particular, three-dimensional AdS black holes [1–3] have been explored to study the emergence of quantum gravitational effects in a simpler setting than the four-dimensional case (see, e.g., refs. [4–6]), and eternal Schwarzschild-AdS black holes have played an important role in the recently discovered relation between the entanglement of quantum states and the Einstein-Rosen non-traversable wormholes [7–12].

Another important reason for the interest in anti-de Sitter black holes is the well-known AdS/CFT correspondence [13–16], which affirms that AdS black holes are dual to equilibrium thermal states of a large N strongly coupled conformal field theory (CFT) on the boundary of the AdS spacetime. In such a context, perturbations of a black hole are associated to small deviations from equilibrium of the CFT thermal system, and the dynamics of linear fluctuations in the dual field theory can be directly studied from the perturbation equations for classical fields in a black-hole spacetime. The AdS/CFT duality at nonzero temperature has become now an important tool to investigate in- and out-of-equilibrium properties of quantum field theories at strong coupling. As a consequence of it, the study of AdS black holes have encountered applications that range from QCD to condensed-matter physics (for reviews, see refs. [17–24]).

The rich structure of the anti-de Sitter spacetime has also been uncovered and has become manifest with the advance of applications of the AdS/CFT correspondence. By itself, the existence of a cosmological constant changes the spacetime asymptotic behavior and also determines the different topologies that a black hole may have. For the case of asymptotically flat four-dimensional spacetimes, Hawking’s topology theorem assures that, under certain reasonable conditions, the horizon of a black hole must be topologically spherical [25, 26]. However, the presence of a negative cosmological constant renders possible the existence of a multiply connected spacetime with an event horizon. A specific identification of points in a planar Reissner-Nordström-AdS black hole [27, 28] generates a charged black string (also called cylindrical charged black hole) or a charged black torus [29], which in turn can be put to rotate through an improper coordinate transformation in the sense of Stachel [30], giving rise to a rotating charged black string or black torus [31, 32].

The perturbations of static charged anti-de Sitter black holes have been the focus of great interest in the last years, especially due to their applications to holographic condensed matter systems (see refs. [33–42] for a sample). In a similar way, rotating uncharged black holes (either in asymptotically flat or AdS spacetimes) have attracted a lot of attention in connection with the Kerr/CFT correspondence [43–47] (see also [48] and references therein). On the other hand, while the geometric and thermodynamic equilibrium properties of rotating charged black holes have been well studied since the discovery of the Kerr-Newman solution [49], the decoupling of the gravitational and

electromagnetic perturbation equations for these black holes remains as a long-standing open problem in general relativity theory [50–52].

Motivated by the correspondence between the near-extremal Kerr-Newman solution and conformal field theories [53, 54], recently there has been a renewed interest in testing the stability of rotating charged black holes against gravitoelectromagnetic fluctuations. This issue has been attacked in different ways, which include: the study of the weak charge limit [55] and the slow rotation limit [56, 57] with perturbative expansions, respectively, around the Kerr and Reissner-Nordström solutions; the computation of the black-hole quasinormal mode spectrum of frequencies by solving a coupled system of perturbation equations [58]; and the performance of numerical simulations in the full Einstein-Maxwell theory [59].

The aforementioned studies practically settled the question of the linear modal stability of the Kerr-Newman black hole and gave strong evidence in favor of its stability beyond the linear level. However, the computation of quantities like scattering amplitudes, greybody factors and correlation functions in the dual CFT requires, in general, the separability of the perturbation equations. In the case of charged black strings, the fact that rotation is implemented via an ‘illegitimate’ boost in the compact direction is a key ingredient for the obtaining of decoupled equations. Hence, the study of rotating charged black strings offers a valuable opportunity to investigate, in a unified way, how the parameters of a black hole (mass, charge and angular momentum) and the global properties of a spacetime (cosmological constant and topology) affect the dynamics of a gravitoelectromagnetic perturbation.

In this paper, we investigate the first-order coupled gravitational and electromagnetic fluctuations of four-dimensional rotating charged black strings in the presence of sources. We generalize here the Chandrasekhar transformation theory [60–62] with the inclusion of source terms, and by combining it with the linearized Newman-Penrose (NP) equations [63], we are able to reduce the perturbation problem to four separated, decoupled inhomogeneous equations of Schrödinger-like form. The set of effective potentials appearing in such equations forms two pairs of supersymmetric partner potentials. Among other things, this implies that the fundamental variables governing one sector of perturbations are related to the variables of the other sector.

The layout of the present article is the following. In the next section, the main geometric properties of a rotating charged black string, and its description in terms of the variables of the Newman-Penrose formalism are reviewed. Section 3 contains the core of the perturbation theory for the charged rotating black string, with the basic set of equations in subsection 3.1, the separation of variables in subsection 3.2, the system of equations for the tetrad invariant NP scalars in 3.3, and the decoupling of the perturbation equations in 3.4. Section 4 is dedicated to transform a pair of decoupled complex equations into four real Schrödinger-like wave equations by means of a generalized Chandrasekhar transformation theory, developed in Appendix C. Section 5 is devoted to investigate the emergence of a quantum-mechanical supersymmetry in the fundamental equations of the black-string perturbation theory. Finally, in section 6

we present a summary of the results and conclude.

2. Rotating charged black strings

2.1. The background spacetime

We consider in this work a four-dimensional Einstein-Maxwell theory with a negative cosmological constant $\Lambda_c = -3/\ell^2$, whose action takes the form

$$I = -\frac{1}{16\pi G} \int d^4x \sqrt{-g} \left(R + \frac{6}{\ell^2} - \ell^2 F_{\mu\nu} F^{\mu\nu} \right), \quad (1)$$

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ and the Greek indices (μ, ν, \dots) run over all the spacetime dimensions. The resulting equations of motion admit a family of asymptotically AdS stationary solutions [32] given by the metric

$$ds^2 = -\frac{r^2}{\ell^2} \gamma^2 f (dt - a d\varphi)^2 + \frac{r^2}{\ell^2} \gamma^2 \left(\ell d\varphi - \frac{a}{\ell} dt \right)^2 + \frac{r^2}{\ell^2} dz^2 + \frac{\ell^2 dr^2}{r^2 f}, \quad (2)$$

and by the gauge potential

$$A = -\gamma \frac{c}{r} (dt - a d\varphi), \quad (3)$$

where

$$f(r) = 1 - \frac{\ell^3 b}{r^3} + \frac{\ell^4 c^2}{r^4} \quad \text{and} \quad \frac{1}{\gamma} = \sqrt{1 - \frac{a^2}{\ell^2}}. \quad (4)$$

If the parameters satisfy $b \geq b_{crit} = 4(c^2/3)^{3/4}$ and the φ - z surfaces have cylindrical topology $(S^1 \times \mathbb{R})$, the metric (2) describes the spacetime geometry of a rotating charged black string with the inner and outer horizon radius, r_- and r_+ , given by the real roots of $f(r) = 0$. For $b = b_{crit}$, the horizons coalesce and we have an extreme black string. The parameters a , b and c can be written in terms of the conserved mass M , angular momentum J and charge Q per unit of black-string length as [32, 64]

$$\begin{aligned} a &= -\frac{3\ell^2}{2J} (\mathcal{Z} - M), & b &= 2G(3\mathcal{Z} - M), \\ c &= 2Q\ell \sqrt{\frac{3\mathcal{Z} - M}{\mathcal{Z} + M}}, & \mathcal{Z} &= \sqrt{M^2 - \frac{8J^2}{9\ell^2}}. \end{aligned} \quad (5)$$

Parameter a is the analogous of the Kerr rotation parameter, representing the angular momentum per unit mass of the black string. The extremal rotation limit in which $J^2/\ell^2 M^2 \rightarrow 1$ or, equivalently, the limit $\mathcal{Z} \rightarrow M/3$, implies in $a = \ell$. On the other hand, the limit of zero angular momentum $J = 0$, or, equivalently, the limit $\mathcal{Z} \rightarrow M$, implies in $a = 0$.

2.2. Description in the Newman-Penrose formalism

For the sequence of the work, it is important to describe the rotating charged black strings in the Newman-Penrose formalism [63], see also Appendix A. The spacetime (2)

is algebraically type D in the Petrov classification. So it is convenient to define the Newman-Penrose (NP) quantities in terms of a Kinnersley-like null frame [65]:

$$D = l^\mu \partial_\mu = \frac{1}{r^2 f} (\ell^2 \gamma \partial_t + a \gamma \partial_\varphi + r^2 f \partial_r), \quad (6)$$

$$\Delta = n^\mu \partial_\mu = \frac{1}{2\ell^2} (\ell^2 \gamma \partial_t + a \gamma \partial_\varphi - r^2 f \partial_r), \quad (7)$$

$$\delta = m^\mu \partial_\mu = \frac{1}{r\sqrt{2}} (\gamma \partial_\varphi + a \gamma \partial_t + i\ell \partial_z), \quad (8)$$

where l^μ and n^μ are the double principal null directions of the Weyl tensor. With this choice of basis vectors, the only non-vanishing spin coefficients are

$$\rho = -\frac{1}{r}, \quad \mu = -\frac{rf}{2\ell^2}, \quad \gamma = \frac{1}{4\ell^2} \frac{d}{dr}(r^2 f). \quad (9)$$

The fact that in the chosen null frame all of the other spin coefficients are identically zero, and particularly $\kappa = \sigma = \lambda = \nu = 0$, confirms the type-D character of the spacetime (2). In accordance with the Goldberg-Sachs theorem [66], the Weyl scalars Ψ_0 , Ψ_1 , Ψ_3 and Ψ_4 vanish, and a direct computation shows that

$$\Psi_2 = C_{\mu\nu\rho\sigma} l^\mu m^\nu m^{*\rho} n^\sigma = -\frac{\ell(br - 2\ell c^2)}{2r^4}, \quad (10)$$

where the asterisk denotes complex conjugation.

In the Newman-Penrose formalism, the electromagnetic field is described by three complex Maxwell scalars: ϕ_0 , ϕ_1 and ϕ_2 . For a rotating charged black string, $\phi_0 = \phi_2 = 0$ and

$$\phi_1 = \frac{1}{2} F_{\mu\nu} (l^\mu n^\nu + m^{*\mu} m^\nu) = \frac{c}{2r^2}. \quad (11)$$

3. Gravitational and electromagnetic perturbations

For the present analysis of the gravitoelectromagnetic perturbations of rotating charged black strings via NP formalism we follow a similar procedure as that presented in Ref. [62] (see also Appendix A for notation and sign conventions). In such an approach the metric and gauge-field linear perturbations are related to infinitesimal changes in the null tetrad vectors and Maxwell scalars, which accordingly lead to first-order perturbations in the spin coefficients and curvature scalars. Among the complete set of NP quantities, we will be mainly interested in the scalars that govern the evolution of the coupled gravitational and electromagnetic waves in the background spacetime (2).

3.1. The set of basic equations

In the study of gravitational perturbations of electrically neutral black holes [67, 68], an important fact to simplify the problem is the linearity and homogeneity of a set of NP equations (four of the Bianchi identities and two of the Ricci identities) in quantities

that vanish in the background spacetime. However, when investigating charged black hole perturbations, we must also consider the Maxwell equations with sources:

$$(\delta^* + \pi - 2\alpha)\phi_0 - (D - 2\rho)\phi_1 - \kappa\phi_2 = 2\pi J_l, \quad (12)$$

$$(\Delta + 2\mu)\phi_1 - (\delta - \tau + 2\beta)\phi_2 - \nu\phi_0 = 2\pi J_n, \quad (13)$$

$$(\Delta + \mu - 2\gamma)\phi_0 - (\delta - 2\tau)\phi_1 - \sigma\phi_2 = 2\pi J_m, \quad (14)$$

$$(\delta^* + 2\pi)\phi_1 - (D - \rho + 2\varepsilon)\phi_2 - \lambda\phi_0 = 2\pi J_{m^*}. \quad (15)$$

Note that these equations contain terms that involve directional derivatives of the scalar ϕ_1 , like $D\phi_1$, $\delta\phi_1$ and so on. Since ϕ_1 is non-vanishing in the background spacetime, a direct linearization of Maxwell equations would give rise to perturbations in the basis vectors (and also in the spin coefficients ρ , τ , π and μ), making difficult the decoupling of the resulting equations.

An alternative to overcome this problem, which was used with success in the Reissner-Nordström black-hole case [50], is looking for another set of equations which are already linearized, in the sense that they are linear and homogeneous in quantities that vanish in the background geometry. By taking into account the presence of source terms, we generalize below the procedure employed by Chandrasekhar [62] in the study of the gravitoelectromagnetic perturbations of Reissner-Nordström black holes.

Initially we apply the operator $(\delta - 2\tau - \alpha^* - \beta + \pi^*)$ to the Maxwell equation (12) and the operator $(D - \varepsilon + \varepsilon^* - 2\rho - \rho^*)$ to the Maxwell equation (14) and subtract one equation from the other to get

$$\begin{aligned} & [(\delta - 2\tau - \alpha^* - \beta + \pi^*)(\delta^* + \pi - 2\alpha) - (D - \varepsilon + \varepsilon^* - 2\rho - \rho^*)(\Delta + \mu - 2\gamma)]\phi_0 = \\ & [(\delta - 2\tau - \alpha^* - \beta + \pi^*)\kappa - (D - \varepsilon + \varepsilon^* - 2\rho - \rho^*)\sigma]\phi_2 + \kappa\delta\phi_2 - \sigma D\phi_2 \\ & + (\delta D - D\delta)\phi_1 + [(\varepsilon - \varepsilon^* + \rho^*)(\delta - 2\tau) - (\alpha^* + \beta - \pi^*)(D - 2\rho)]\phi_1 \\ & - 2(\delta\rho)\phi_1 + 2(D\tau)\phi_1 + 2\pi[(\delta - 2\tau - \alpha^* - \beta + \pi^*)J_l - (D - \varepsilon + \varepsilon^* - 2\rho - \rho^*)J_m]. \end{aligned} \quad (16)$$

To simplify the resulting equation, we utilize the commutation relation (A.8) and the Ricci identities (A.22) and (A.25) to eliminate, respectively, the operator $(\delta D - D\delta)$ and the quantities $D\tau$ and $\delta\rho$ from (16). After some manipulations, it results in the following equation:

$$\begin{aligned} & [(\delta - 2\tau - \alpha^* - \beta + \pi^*)(\delta^* + \pi - 2\alpha) - (D - \varepsilon + \varepsilon^* - 2\rho - \rho^*)(\Delta + \mu - 2\gamma)]\phi_0 = \\ & [(\delta - 2\tau - \alpha^* - \beta + \pi^*)\kappa - (D - \varepsilon + \varepsilon^* - 2\rho - \rho^*)\sigma]\phi_2 + \kappa\delta\phi_2 - \sigma D\phi_2 \\ & + 2\phi_1 [(\Delta - 3\gamma - \gamma^* - \mu + \mu^*)\kappa - (\delta^* - 3\alpha + \beta^* - \tau^* - \pi)\sigma] + 4\Psi_1\phi_1 + \kappa\Delta\phi_1 \\ & - \sigma\delta^*\phi_1 + 2\pi[(\delta - 2\tau - \alpha^* - \beta + \pi^*)J_l - (D - \varepsilon + \varepsilon^* - 2\rho - \rho^*)J_m]. \end{aligned} \quad (17)$$

In a similar way, applying the operator $(\delta^* - \tau^* + \alpha + \beta^* + 2\pi)$ to equation (13) and the operator $(\Delta + \mu^* - \gamma^* + \gamma + 2\mu)$ to equation (15) and subtracting one of the resulting

equations from the other, we obtain

$$\begin{aligned}
& [(\delta^* - \tau^* + \alpha + \beta^* + 2\pi)(\delta - \tau + 2\beta) - (\Delta + \mu^* - \gamma^* + \gamma + 2\mu)(D - \rho + 2\varepsilon)] \phi_2 = \\
& - [(\delta^* - \tau^* + \alpha + \beta^* + 2\pi)\nu - (\Delta + \mu^* - \gamma^* + \gamma + 2\mu)\lambda] \phi_0 - \nu\delta^*\phi_0 + \lambda\Delta\phi_0 \\
& + (\delta^*\Delta - \Delta\delta^*)\phi_1 - [(\mu^* - \gamma^* + \gamma)(\delta^* + 2\pi) + (\tau^* - \alpha - \beta^*)(\Delta + 2\mu)] \phi_1 + 2(\delta^*\mu)\phi_1 \\
& - 2(\Delta\pi)\phi_1 - 2\pi[(\delta^* - \tau^* + \alpha + \beta^* + 2\pi)J_n - (\Delta + \mu^* - \gamma^* + \gamma + 2\mu)J_{m^*}].
\end{aligned} \tag{18}$$

Then, using the complex conjugate of the commutation relation (A.9) and the Ricci identities (A.23) and (A.26) to eliminate, respectively, the operator $(\delta^*\Delta - \Delta\delta^*)$ and the quantities $\Delta\pi$ and $\delta^*\mu$ from (18), we find

$$\begin{aligned}
& [(\delta^* - \tau^* + \alpha + \beta^* + 2\pi)(\delta - \tau + 2\beta) - (\Delta + \mu^* - \gamma^* + \gamma + 2\mu)(D - \rho + 2\varepsilon)] \phi_2 = \\
& - [(\delta^* - \tau^* + \alpha + \beta^* + 2\pi)\nu - (\Delta + \mu^* - \gamma^* + \gamma + 2\mu)\lambda] \phi_0 + \lambda\Delta\phi_0 - \nu\delta^*\phi_0 \\
& - 2\phi_1 [(D + 3\varepsilon + \varepsilon^* + \rho - \rho^*)\nu - (\delta + \pi^* + \tau - \alpha^* + 3\beta)\lambda] + 4\Psi_3\phi_1 - \nu D\phi_1 \\
& + \lambda\delta\phi_1 - 2\pi[(\delta^* - \tau^* + \alpha + \beta^* + 2\pi)J_n - (\Delta + \mu^* - \gamma^* + \gamma + 2\mu)J_{m^*}].
\end{aligned} \tag{19}$$

Equations (17) and (19) are already linearized in the sense that they give rise to equations which are linear and homogeneous in the quantities that vanish in the background. In fact, the terms involving ϕ_2 in (17) and the terms involving ϕ_0 in (19) consist of quantities of second order, and so they can be ignored in a linear perturbation theory. On basis of the Maxwell equations (12)-(15) for the background spacetime, the terms $\Delta\phi_1$, $\delta^*\phi_1$, $D\phi_1$ and $\delta\phi_1$ can be replaced, respectively, by $-2\mu\phi_1$, $-2\pi\phi_1$, $2\rho\phi_1$ and $2\tau\phi_1$. As a result, the linearized versions of equations (17) and (19) become

$$\begin{aligned}
& [\delta\delta^* - (D - 2\rho - \rho^*)(\Delta + \mu - 2\gamma)]\phi_0^{(1)} - 2\phi_1 [(\Delta - 3\gamma - \gamma^* - 2\mu + \mu^*)\kappa^{(1)} \\
& - \delta^*\sigma^{(1)} + 2\Psi_1^{(1)}] = 2\pi[\delta J_l - (D - 2\rho - \rho^*)J_m],
\end{aligned} \tag{20}$$

$$\begin{aligned}
& [\delta^*\delta - (\Delta + \mu^* - \gamma^* + \gamma + 2\mu)(D - \rho)]\phi_2^{(1)} - 2\phi_1[\delta\lambda^{(1)} - (D + 2\rho - \rho^*)\nu^{(1)} \\
& + 2\Psi_3^{(1)}] = 2\pi[(\Delta + 3\mu)J_{m^*} - \delta^*J_n],
\end{aligned} \tag{21}$$

where the superscript (1) is used to distinguish the first-order perturbation of a NP quantity from its value in the stationary background state, and the vanishing of some of the spin coefficients in the background spacetime was used to simplify the above equations.

In addition to the linearized equations (20) and (21), it is important to consider here the pair of Ricci equations (A.21) and (A.24) and the set of Bianchi identities (A.39)-(A.42). The main difference of these equations in relation to the electrically neutral black hole case is the appearance of Ricci (source) terms in the Bianchi identities, which depend on the square of the Maxwell scalars and arise due to the coupling between the metric and gauge field.

The linearized Bianchi identities (A.39)-(A.42), governing the radiative (nontrivial) parts of the black-string perturbations, take respectively the forms

$$\begin{aligned}
& \delta^*\Psi_0^{(1)} - (D - 4\rho)\Psi_1^{(1)} + 2\ell^2\phi_1^*D\phi_0^{(1)} - (3\Psi_2 - 4\ell^2\phi_1\phi_1^*)\kappa^{(1)} = \\
& 4\pi G [-(D - 2\rho)T_{lm}^{(\text{MAT})} + \delta T_{ll}^{(\text{MAT})}],
\end{aligned} \tag{22}$$

$$(\Delta - 4\gamma + \mu)\Psi_0^{(1)} - \delta\Psi_1^{(1)} - 2\ell^2\phi_1^*\delta\phi_0^{(1)} - (3\Psi_2 + 4\ell^2\phi_1\phi_1^*)\sigma^{(1)} = 4\pi G \left[-(D + \rho^*)T_{mm}^{(\text{MAT})} + \delta T_{lm}^{(\text{MAT})} \right], \quad (23)$$

$$\delta\Psi_4^{(1)} - (\Delta + 2\gamma + 4\mu)\Psi_3^{(1)} + 2\ell^2\phi_1^*(\Delta + 2\gamma)\phi_2^{(1)} + (3\Psi_2 - 4\ell^2\phi_1\phi_1^*)\nu^{(1)} = 4\pi G \left[(\Delta + 2\mu^* + 2\gamma)T_{nm^*}^{(\text{MAT})} - \delta^*T_{nn}^{(\text{MAT})} \right], \quad (24)$$

$$(D - \rho)\Psi_4^{(1)} - \delta^*\Psi_3^{(1)} - 2\ell^2\phi_1^*\delta^*\phi_2^{(1)} + (3\Psi_2 + 4\ell^2\phi_1\phi_1^*)\lambda^{(1)} = 4\pi G \left[-(\Delta + \mu^*)T_{m^*m^*}^{(\text{MAT})} + \delta^*T_{nm^*}^{(\text{MAT})} \right], \quad (25)$$

where $T_{ab}^{(\text{MAT})}$ ($a, b = l, n, m, m^*$) are the tetrad components of the energy-momentum tensor of all forms of matter and all nonelectromagnetic and nongravitational fields. The above equations are supplemented by the following linearized Ricci identities:

$$(D - \rho - \rho^*)\sigma^{(1)} - \delta\kappa^{(1)} = \Psi_0^{(1)}, \quad (26)$$

$$(\Delta + \mu + \mu^* + 3\gamma - \gamma^*)\lambda^{(1)} - \delta^*\nu^{(1)} = \Psi_4^{(1)}. \quad (27)$$

The set of equations (20)-(27) forms the basic set of equations for the study of the gravitoelectromagnetic perturbations of the rotating charged black strings.

3.2. Separation of variables

As usual we take the Fourier transform of the perturbation functions, i.e., we assume a dependence on the coordinates t , φ and z of the form $\exp[-i\omega t + im\varphi + ikz]$, where m is an integer number. So the action of the directional derivatives D , Δ , δ and δ^* on functions with such an exponential dependence become

$$D = \mathcal{D}_0, \quad \Delta = -\frac{r^2 f}{2\ell^2} \mathcal{D}_0^\dagger, \quad \delta = \frac{i}{r\sqrt{2}} p, \quad \delta^* = \frac{i}{r\sqrt{2}} p^*, \quad (28)$$

where

$$\mathcal{D}_n = \frac{d}{dr} - \frac{i\ell^2}{r^2 f} \varpi + n \frac{d}{dr} \ln \left(\frac{r^4}{\ell^2} f \right), \quad \mathcal{D}_n^\dagger = \frac{d}{dr} + \frac{i\ell^2}{r^2 f} \varpi + n \frac{d}{dr} \ln \left(\frac{r^4}{\ell^2} f \right), \quad (29)$$

and the constants p and ϖ are given by

$$p = \frac{\gamma}{\ell} (m - a\omega) + ik, \quad \varpi = \gamma \left(\omega - \frac{am}{\ell^2} \right). \quad (30)$$

The differential operators \mathcal{D}_n and \mathcal{D}_n^\dagger satisfy some identities that will be useful in the sequence of this work:

$$(\mathcal{D}_n)^* = \mathcal{D}_n^\dagger, \quad \mathcal{D}_n \left[r^{N'} \left(\frac{r^4}{\ell^2} f \right)^N \right] = r^{N'} \left(\frac{r^4}{\ell^2} f \right)^N \left[\mathcal{D}_{(n+N)} + \frac{N'}{r} \right], \quad (31)$$

where n , N and N' are integer numbers.

We work from now on with the Fourier transformed versions of equations (20)-(27), but keep the same symbols for the Fourier-transforms as the original NP quantities.

Substituting the background values (9)-(11) for the spin coefficients ρ , γ and μ , for the Weyl scalar Ψ_2 and the Maxwell scalar ϕ_1 , equations (20)-(27) become

$$\begin{aligned}
& -\frac{ip^*}{\sqrt{2}r}\Psi_0^{(1)} + \left(\mathcal{D}_0 + \frac{4}{r}\right)\Psi_1^{(1)} - \frac{c\ell^2}{r^2}\mathcal{D}_0\phi_0^{(1)} - \frac{(3\ell br - 4\ell^2 c^2)}{2r^4}\kappa^{(1)} = \frac{\mathcal{T}_\mathcal{A}}{\sqrt{2}r}, \\
& \frac{r^2 f}{2\ell^2}\left(\mathcal{D}_2^\dagger - \frac{3}{r}\right)\Psi_0^{(1)} + \frac{ip}{\sqrt{2}r}\left(\Psi_1^{(1)} + \frac{c\ell^2}{r^2}\phi_0^{(1)}\right) - \frac{(3\ell br - 8\ell^2 c^2)}{2r^4}\sigma^{(1)} = \frac{\mathcal{T}_\mathcal{B}}{2r^2}, \\
& \left(\mathcal{D}_0 + \frac{2}{r}\right)\sigma^{(1)} - \frac{ip}{\sqrt{2}r}\kappa^{(1)} - \Psi_0^{(1)} = 0, \\
& \frac{r^2 f}{2\ell^2}\left(\mathcal{D}_2^\dagger - \frac{5}{r}\right)\kappa^{(1)} + \frac{ip^*}{\sqrt{2}r}\sigma^{(1)} + \left[\frac{r^4 f}{\ell^2}\left(\mathcal{D}_1 + \frac{1}{r}\right)\left(\mathcal{D}_1^\dagger - \frac{1}{r}\right) - p^2\right]\frac{\phi_0^{(1)}}{2c} - 2\Psi_1^{(1)} = \frac{\mathcal{J}_\mathcal{B}}{\sqrt{2}},
\end{aligned} \tag{32}$$

and

$$\begin{aligned}
& \frac{ip}{\sqrt{2}r}\Psi_4^{(1)} + \frac{r^2 f}{2\ell^2}\left(\mathcal{D}_{-1}^\dagger + \frac{6}{r}\right)\Psi_3^{(1)} - \frac{cf}{2}\left(\mathcal{D}_{-1}^\dagger + \frac{2}{r}\right)\phi_2^{(1)} - \frac{(3\ell br - 4\ell^2 c^2)}{2r^4}\nu^{(1)} = \frac{\mathcal{T}_\mathcal{C}}{\sqrt{2}r^5}, \\
& \left(\mathcal{D}_0 + \frac{1}{r}\right)\Psi_4^{(1)} - \frac{ip^*}{\sqrt{2}r}\left(\Psi_3^{(1)} + \frac{c\ell^2}{r^2}\phi_2^{(1)}\right) - \frac{(3\ell br - 8\ell^2 c^2)}{2r^4}\lambda^{(1)} = \frac{\mathcal{T}_\mathcal{D}}{r^4}, \\
& \frac{r^2}{2\ell^2}f\left(\mathcal{D}_{-1}^\dagger + \frac{4}{r}\right)\lambda^{(1)} + \frac{ip^*}{\sqrt{2}r}\nu^{(1)} - \Psi_4^{(1)} = 0, \\
& -\left(\mathcal{D}_0 - \frac{1}{r}\right)\nu^{(1)} + \frac{ip}{\sqrt{2}r}\lambda^{(1)} - \left[\frac{r^4 f}{\ell^2}\left(\mathcal{D}_0^\dagger + \frac{3}{r}\right)\left(\mathcal{D}_0 + \frac{1}{r}\right) - p^2\right]\frac{\phi_2^{(1)}}{2c} + 2\Psi_3^{(1)} = \frac{\sqrt{2}}{r^2}\mathcal{J}_\mathcal{D},
\end{aligned} \tag{33}$$

where the source terms in the above equations are given by

$$\mathcal{T}_\mathcal{A} = 4\pi G \left[\sqrt{2}r \left(\mathcal{D}_0 + \frac{2}{r} \right) T_{lm}^{(\text{MAT})} - ip T_{ll}^{(\text{MAT})} \right], \tag{34}$$

$$\mathcal{T}_\mathcal{B} = 4\sqrt{2}\pi G r \left[\sqrt{2}r \left(\mathcal{D}_0 + \frac{1}{r} \right) T_{mm}^{(\text{MAT})} - ip T_{lm}^{(\text{MAT})} \right], \tag{35}$$

$$\mathcal{J}_\mathcal{B} = \frac{2\pi r}{c} \left[ip J_l - \sqrt{2}r \left(\mathcal{D}_0 + \frac{3}{r} \right) J_m \right], \tag{36}$$

$$\mathcal{T}_\mathcal{C} = -4\pi G r^4 \left[\frac{r^3 f}{\sqrt{2}\ell^2} \left(\mathcal{D}_{-1}^\dagger + \frac{4}{r} \right) T_{nm}^{(\text{MAT})} + ip^* T_{nn}^{(\text{MAT})} \right], \tag{37}$$

$$\mathcal{T}_\mathcal{D} = 2\sqrt{2}\pi G r^3 \left[\frac{r^3 f}{\sqrt{2}\ell^2} \left(\mathcal{D}_0^\dagger + \frac{1}{r} \right) T_{m^*m^*}^{(\text{MAT})} + ip T_{nm}^{(\text{MAT})} \right], \tag{38}$$

$$\mathcal{J}_\mathcal{D} = \frac{\pi r^3}{c} \left[ip^* J_n + \frac{r^3 f}{\sqrt{2}\ell^2} \left(\mathcal{D}_0^\dagger + \frac{3}{r} \right) J_{m^*} \right], \tag{39}$$

and the subscripts \mathcal{A} , \mathcal{B} , \mathcal{C} and \mathcal{D} are used only to distinguish the different source terms.

3.3. Equations for tetrad-invariant scalars

An important issue in a perturbation theory is the invariance of the basic quantities under general coordinate transformations and infinitesimal gauge transformations

respectively in the coordinates and null basis vectors. Being scalar functions with vanishing background values, the variables $\Psi_0, \Psi_1, \Psi_3, \Psi_4, \phi_0, \phi_2$ and the spin coefficients σ, κ, λ and ν are invariant under *gauge transformations of the first and second kind* [69], i.e., they are independent of the choice of the coordinate system on the physical manifold and independent of the identification map between points of the fictitious background spacetime and the physical spacetime.

As discussed in detail in Appendix B, there are also gauge degrees of freedom associated to infinitesimal transformations on the tetrad vectors. The basic NP variables appearing in equations (32) and (33) are invariant under transformations of class III, but change under null rotations of class I and II, as shown in equations (B.6), (B.7) and (B.9). However, it is easy to verify that the following combinations of NP quantities

$$\begin{aligned} \Psi_0^{(1)}, \quad \Psi_1^{(1)} - \frac{3}{2} \left(\frac{\Psi_2}{\phi_1} \right) \phi_0^{(1)}, \quad \kappa^{(1)} + \frac{1}{2} (D - \rho) \frac{\phi_0^{(1)}}{\phi_1}, \quad \sigma^{(1)} + \frac{1}{2} \delta \frac{\phi_0^{(1)}}{\phi_1}, \\ \Psi_3^{(1)} - \frac{3}{2} \left(\frac{\Psi_2}{\phi_1} \right) \phi_2^{(1)}, \quad \Psi_4^{(1)}, \quad \nu^{(1)} - \frac{1}{2} (\Delta + \mu + 2\gamma) \frac{\phi_2^{(1)}}{\phi_1}, \quad \lambda^{(1)} - \frac{1}{2} \delta^* \frac{\phi_2^{(1)}}{\phi_1}, \end{aligned} \quad (40)$$

are invariant under infinitesimal rotations of the null basis. The foregoing variables constitute a set of tetrad and coordinate-gauge independent quantities governing the gravitoelectromagnetic perturbations of rotating charged black strings.

Equations (32) and (33) take particularly simple and symmetrical forms with the introduction of the following new variables

$$\begin{aligned} \Phi_0 = \Psi_0^{(1)}, \quad \Phi_1 = r\sqrt{2} \left[\Psi_1^{(1)} + \frac{3}{cr} \left(\frac{b\ell}{2} - \frac{c^2\ell^2}{r} \right) \phi_0^{(1)} \right], \\ \mathcal{K} = \frac{1}{\sqrt{2}} \left[\frac{\kappa^{(1)}}{r^2} + \frac{1}{c} \left(\mathcal{D}_0 + \frac{3}{r} \right) \phi_0^{(1)} \right], \quad \mathcal{S} = \frac{1}{r} \left(\sigma^{(1)} + \frac{r}{\sqrt{2}c} ip \phi_0^{(1)} \right), \end{aligned} \quad (41)$$

and

$$\begin{aligned} \Phi_3 = \frac{r^3}{\sqrt{2}} \left[\Psi_3^{(1)} + \frac{3}{cr} \left(\frac{b\ell}{2} - \frac{c^2\ell^2}{r} \right) \phi_2^{(1)} \right], \quad \Phi_4 = r^4 \Psi_4^{(1)}, \\ \mathcal{N} = \frac{r^2}{\sqrt{2}} \left[\nu^{(1)} + \frac{r^4 f}{2c\ell^2} \left(\mathcal{D}_{-1}^\dagger + \frac{5}{r} \right) \phi_2^{(1)} \right], \quad \mathcal{L} = \frac{1}{2} r \left(\lambda^{(1)} - \frac{r}{\sqrt{2}c} ip^* \phi_2^{(1)} \right), \end{aligned} \quad (42)$$

which are proportional to the Fourier transforms of the quantities (40). This replacement of variables leads to the following set of perturbation equations:

$$-ip^* \Phi_0 + \left(\mathcal{D}_0 + \frac{3}{r} \right) \Phi_1 - \left(3\ell b - \frac{4\ell^2 c^2}{r} \right) \mathcal{K} = \mathcal{T}_A, \quad (43)$$

$$\frac{r^4 f}{\ell^2} \left(\mathcal{D}_2^\dagger - \frac{3}{r} \right) \Phi_0 + ip \Phi_1 - \left(3\ell b - \frac{8\ell^2 c^2}{r} \right) \mathcal{S} = \mathcal{T}_B, \quad (44)$$

$$\left(\mathcal{D}_0 + \frac{3}{r} \right) \mathcal{S} - ip \mathcal{K} - \frac{\Phi_0}{r} = 0, \quad (45)$$

$$\frac{r^4 f}{\ell^2} \left(\mathcal{D}_2^\dagger - \frac{3}{r} \right) \mathcal{K} + ip^* \mathcal{S} - 2 \frac{\Phi_1}{r} = \mathcal{J}_B, \quad (46)$$

and

$$ip \Phi_4 + \frac{r^4 f}{\ell^2} \left(\mathcal{D}_{-1}^\dagger + \frac{3}{r} \right) \Phi_3 - \left(3\ell b - \frac{4\ell^2 c^2}{r} \right) \mathcal{N} = \mathcal{T}_c, \quad (47)$$

$$\left(\mathcal{D}_0 - \frac{3}{r} \right) \Phi_4 - ip^* \Phi_3 - \left(3\ell b - \frac{8\ell^2 c^2}{r} \right) \mathcal{L} = \mathcal{T}_D, \quad (48)$$

$$\frac{r^4 f}{\ell^2} \left(\mathcal{D}_{-1}^\dagger + \frac{3}{r} \right) \mathcal{L} + ip^* \mathcal{N} - \frac{\Phi_4}{r} = 0, \quad (49)$$

$$- \left(\mathcal{D}_0 - \frac{3}{r} \right) \mathcal{N} + ip \mathcal{L} + 2 \frac{\Phi_3}{r} = \mathcal{J}_D, \quad (50)$$

where, to obtain these final relations, we have made use of the second one of the identities given in equation (31).

3.4. Decoupling of the perturbation equations

The set of perturbation equations (43)-(50) include two decoupled systems of equations: (43)-(46) govern the evolution of Φ_0 , Φ_1 , \mathcal{K} and \mathcal{S} , and (47)-(50) govern the scalars Φ_3 , Φ_4 , \mathcal{L} and \mathcal{N} . In the following, we study first the set of equations (43)-(46). The decoupling of these equations and their subsequent reduction to a pair of second-order differential equations can be carried out with the introduction of the functions

$$\begin{aligned} \mathcal{F}_{+1} &= \Phi_0 - i \frac{q_1}{p^*} \mathcal{K}, & \mathcal{G}_{+1} &= \Phi_1 + i \frac{q_1}{p} \mathcal{S}, \\ \mathcal{F}_{+2} &= \Phi_0 - i \frac{q_2}{p^*} \mathcal{K}, & \mathcal{G}_{+2} &= \Phi_1 + i \frac{q_2}{p} \mathcal{S}, \end{aligned} \quad (51)$$

where q_1 and q_2 are defined as

$$q_1 = \frac{1}{2} \ell \left[3b + \sqrt{9b^2 + 16c^2 p^2} \right], \quad q_2 = \frac{1}{2} \ell \left[3b - \sqrt{9b^2 + 16c^2 p^2} \right], \quad (52)$$

with

$$p^2 = pp^* = \frac{\Upsilon^2}{\ell^2} (m - a\omega)^2 + k^2. \quad (53)$$

The quantities q_1 and q_2 obey the following relations:

$$q_1 + q_2 = 3\ell b \quad \text{and} \quad -q_1 q_2 = 4\ell^2 c^2 p^2. \quad (54)$$

A set of equations governing the evolution of variables \mathcal{F}_{+i} and \mathcal{G}_{+i} ($i = 1, 2$) can be found from suitable combinations of equations (43)-(46). For instance, the addition of equation (44) to equation (46) multiplied by the factor $-iq_1/p^*$ furnishes

$$\frac{r^4}{\ell^2} f \left(\mathcal{D}_2^\dagger - \frac{3}{r} \right) \mathcal{F}_{+1} + ip \left(1 + \frac{2q_1}{p^2 r} \right) \mathcal{G}_{+2} = \mathcal{T}_B - \frac{iq_1}{p^*} \mathcal{J}_B, \quad (55)$$

where, to simplify the above equation, it was made use of relations (54). An analogous combination of equations (44) and (46), with the term $-iq_2/p^*$ in place of $-iq_1/p^*$,

results in a relation of the same form as equation (55), except by the interchange of the indices 1 and 2. Both formulas can be compacted into the single expression

$$\frac{r^4}{\ell^2} f \left(\mathcal{D}_2^\dagger - \frac{3}{r} \right) \mathcal{F}_{+i} + ip \left(1 + \frac{2q_i}{p^2 r} \right) \mathcal{G}_{+j} = \mathcal{T}_\mathcal{B} - \frac{iq_i}{p^*} \mathcal{J}_\mathcal{B}, \quad (i, j = 1, 2; i \neq j). \quad (56)$$

A similar procedure involving equations (43) and (45) leads to

$$\left(\mathcal{D}_0 + \frac{3}{r} \right) \mathcal{G}_{+i} - ip^* \left(1 + \frac{q_i}{p^2 r} \right) \mathcal{F}_{+j} = \mathcal{T}_\mathcal{A}, \quad (i, j = 1, 2; i \neq j). \quad (57)$$

The convention that i and j assume values 1 and 2 with $i \neq j$ will be used from now on in various expressions of this work without explicit indication.

A further simplification of equations (55) and (56) can be performed with the exchange of \mathcal{F}_{+i} and \mathcal{G}_{+i} by the new functions

$$Y_{+i} = \frac{r^5 f^2}{\ell^4} \mathcal{F}_{+i}, \quad X_{+i} = r^3 \mathcal{G}_{+i}. \quad (58)$$

In terms of these variables, equations (56) and (57) become, respectively,

$$\Lambda_+ Y_{+i} + ip \frac{f^2}{\ell^4} \left(1 + \frac{2q_i}{p^2 r} \right) X_{+j} = \frac{r^3 f}{\ell^2} \left(\mathcal{T}_\mathcal{B} - \frac{iq_i}{p^*} \mathcal{J}_\mathcal{B} \right), \quad (59)$$

$$\Lambda_- X_{+j} - ip^* \frac{\ell^2}{f} \left(1 + \frac{q_j}{p^2 r} \right) Y_{+i} = \frac{r^5 f}{\ell^2} \mathcal{T}_\mathcal{A}, \quad (60)$$

where the operators Λ_+ and Λ_- are given by

$$\Lambda_+ = \frac{d}{dr_*} + i\varpi = \frac{r^2}{\ell^2} f \mathcal{D}_0^\dagger, \quad \Lambda_- = \frac{d}{dr_*} - i\varpi = \frac{r^2}{\ell^2} f \mathcal{D}_0, \quad (61)$$

and the tortoise coordinate r_* is defined implicitly by $dr/dr_* = r^2 f / \ell^2$.

Finally, we choose to eliminate the X 's in favor of the Y 's in (59) and (60), and so we obtain the following decoupled equations:

$$\Lambda^2 Y_{+i} + P_i \Lambda_+ Y_{+i} - Q_i Y_{+i} = \mathfrak{S}_{+i}, \quad (62)$$

where

$$\Lambda^2 = \Lambda_+ \Lambda_- = \Lambda_- \Lambda_+ = \frac{d^2}{dr_*^2} - \varpi^2. \quad (63)$$

The functions P_i and Q_i appearing in equation (62) are given by

$$P_i = \frac{d}{dr_*} \ln \left(\frac{r^8}{\mathcal{D}_i} \right), \quad Q_i = p^2 \frac{\ell^2 \mathcal{D}_i}{r^8 f} \left(1 + \frac{q_j}{p^2 r} \right), \quad (64)$$

with

$$\mathcal{D}_i = \frac{r^8 f^2}{\ell^4} \left(1 + \frac{2q_i}{p^2 r} \right), \quad (65)$$

and the source terms \mathfrak{S}_{+i} ($i = 1, 2$) can be written as

$$\mathfrak{S}_{+i} = -ip \frac{f}{r^3 \ell^2} \mathcal{D}_i \mathcal{T}_A + (\Lambda_- + P_i) \frac{r^3 f}{\ell^2} \left(\mathcal{T}_B - \frac{iq_i}{p^*} \mathcal{J}_B \right). \quad (66)$$

The decoupling of the second group of perturbation equations, (47)-(50), can be carried out with the introduction of the functions

$$\begin{aligned} \mathcal{F}_{-1} &= \Phi_4 + i \frac{q_1}{p} \mathcal{N}, & \mathcal{G}_{-1} &= \Phi_3 - i \frac{q_1}{p^*} \mathcal{L}, \\ \mathcal{F}_{-2} &= \Phi_4 + i \frac{q_2}{p} \mathcal{N}, & \mathcal{G}_{-2} &= \Phi_3 - i \frac{q_2}{p^*} \mathcal{L}, \end{aligned} \quad (67)$$

and

$$Y_{-i} = \frac{1}{r^3} \mathcal{F}_{-i}, \quad X_{-i} = \frac{\ell^2}{rf} \mathcal{G}_{-i}. \quad (68)$$

By means of a similar sequence of substitutions and reductions as above, we obtain a pair of decoupled second-order differential equations for Y_{-1} and Y_{-2} :

$$\Lambda^2 Y_{-i} + P_i \Lambda_- Y_{-i} - Q_i Y_{-i} = \mathfrak{S}_{-i}, \quad (69)$$

where the source terms \mathfrak{S}_{-i} ($i = 1, 2$) are given by

$$\mathfrak{S}_{-i} = ip^* \frac{\ell^2}{r^{11} f} \mathcal{D}_i \mathcal{T}_c + (\Lambda_+ + P_i) \frac{f}{r \ell^2} \left(\mathcal{T}_D - \frac{iq_i}{p} \mathcal{J}_D \right). \quad (70)$$

Equations (62) and (69) are then the uncoupled fundamental equations governing the gravitoelectromagnetic perturbations of the rotating charged black strings.

4. Chandrasekhar transformations and the Schrödinger-like equations

Once we have obtained the fundamental perturbation equations for $Y_{\pm i}$, we can now use the generalized Chandrasekhar transformation theory [60–62] of Appendix C to reduce a pair of complex equations (those for Y_{+1} and Y_{+2} , say) to four real one-dimensional Schrödinger-like equations. The condition for the equations (62) or (69) to be transformed into the form

$$\Lambda^2 Z_i^{(\pm)} = V_i^{(\pm)} Z_i^{(\pm)} + \mathcal{F}_i^{(\pm)}, \quad (71)$$

by means of the *dual* transformations of Heading [70], is the existence of suitable constants \mathfrak{B}_i^2 and \varkappa_i , such that the functions

$$F_i = r^8 \frac{Q_i}{\mathcal{D}_i} = \frac{\ell^2}{rf} (p^2 r + q_j) \quad (i, j = 1, 2; i \neq j) \quad (72)$$

satisfy the nonlinear differential equations

$$\frac{1}{F_i} \left(\frac{dF_i}{dr_*} \right)^2 - \frac{d^2 F_i}{dr_*^2} + \frac{\mathcal{D}_i}{r^8} F_i^2 = \frac{\mathfrak{B}_i^2}{F_i} + \varkappa_i. \quad (73)$$

Substituting expression (72) for F_i into the last equation, we obtain the following values for \mathfrak{B}_i^2 and \varkappa_i :

$$\mathfrak{B}_i^2 = q_j^2 \quad (i, j = 1, 2; i \neq j) \quad \text{and} \quad \varkappa_i = p^4. \quad (74)$$

The fact that \mathfrak{B}_i appears in equation (73) as \mathfrak{B}_i^2 gives rise to a pair of dual transformations for each value of i : a transformation associated with $\mathfrak{B}_i^{(+)} = +q_j$ and another one corresponding to $\mathfrak{B}_i^{(-)} = -q_j$. As it is usual in these cases, we use here the superscripts (\pm) to distinguish between transformations involving $+q_j$ and $-q_j$.

From the generalized transformation theory of Appendix C, it follows that the frequency-dependent dual potentials $V_i^{(\pm)}$ for perturbations of a black string are such that

$$V_i^{(\pm)} = \pm q_j \frac{d\mathfrak{f}_i}{dr_*} + q_j^2 \mathfrak{f}_i^2 + p^4 \mathfrak{f}_i, \quad (75)$$

with

$$\mathfrak{f}_i = \frac{1}{F_i} = \frac{rf}{\ell^2(p^2r + q_j)} \quad (i, j = 1, 2; i \neq j). \quad (76)$$

It is not difficult to write the effective potentials (75) in an explicit form. For the sector of perturbations labeled by $(-)$, a direct computation shows that

$$V_i^{(-)} = \frac{f}{\ell^2} \left[p^2 + 4 \frac{c^2 \ell^2}{r^2} - \frac{q_j}{r} \right], \quad (77)$$

while the effective potentials labeled by $(+)$ are given by

$$V_i^{(+)} = \frac{f}{\ell^2(p^2r + q_j)^2} \left[q_j p^2 \left(3b\ell - 2 \frac{c^2 \ell^2}{r} \right) + 2q_j^2 \left(\frac{r^2}{\ell^2} + \frac{b\ell}{2r} - \frac{c^2 \ell^2}{r^2} \right) + p^4 r(p^2r + q_j) \right]. \quad (78)$$

The source terms $\mathfrak{S}_{\pm i}$ and $\mathcal{F}_i^{(\pm)}$, which appear in the differential equations (62), (69) and (71), are related by

$$\mathfrak{S}_{+i} = (P_i + \Lambda_-) (\Xi_i^{(\pm)} + \Lambda_-) \mathcal{F}_i^{(\pm)}, \quad (79)$$

$$\mathfrak{S}_{-i} = (P_i + \Lambda_+) (\Xi_i^{(\pm)} + \Lambda_+) \mathcal{F}_i^{(\pm)}, \quad (80)$$

where the auxiliary functions $\Xi_i^{(-)}$ and $\Xi_i^{(+)}$ are

$$\Xi_i^{(\pm)} = -\frac{d}{dr_*} \ln \mathfrak{f}_i \mp q_j \mathfrak{f}_i, \quad (81)$$

or, explicitly,

$$\Xi_i^{(-)} = \Xi_i^{(-)} = \frac{1}{r^2} \left(-3b\ell + 4 \frac{c^2 \ell^2}{r} \right) \quad (82)$$

and

$$\Xi_i^{(+)} = \Xi_i^{(-)} - 2q_j \frac{rf}{\ell^2(p^2r + q_j)}. \quad (83)$$

According to equations (C.37)-(C.41) of Appendix C, the inverse relations of (79) and (80) can be determined by a pair of consecutive quadratures of the equations

$$\frac{d}{dr_*} \left[h_i^{(\pm)} \frac{d}{dr_*} (g_i^{(\pm)} \mathcal{F}_i^{(\pm)} e^{-i\varpi r_*}) \right] = h_i^{(\pm)} g_i^{(\pm)} \mathfrak{S}_{+i} e^{-i\varpi r_*}, \quad (84)$$

$$\frac{d}{dr_*} \left[h_i^{(\pm)} \frac{d}{dr_*} (g_i^{(\pm)} \mathcal{F}_i^{(\pm)} e^{+i\varpi r_*}) \right] = h_i^{(\pm)} g_i^{(\pm)} \mathfrak{S}_{-i} e^{+i\varpi r_*}. \quad (85)$$

After a convenient choice of integration constant ($C_i^{(\pm)} = 0$) in equations (C.37) and (C.38), the functions $h_i^{(-)}$ and $g_i^{(-)}$ are reduced to

$$h_i^{(-)} = \frac{r^6 f}{\ell^2 \mathcal{D}_i} \left(1 + \frac{q_i}{p^2 r} \right)^2, \quad g_i^{(-)} = \frac{\ell^2 r}{f} \left(1 + \frac{q_i}{p^2 r} \right)^{-1}, \quad (86)$$

while the functions $h_i^{(+)}$ and $g_i^{(+)}$ become

$$h_i^{(+)} = \frac{\ell^2}{f \mathcal{D}_i F_i^2} [b\ell r (3p^2 r + 4q_j) - q_j (2\ell^2 c^2 + q_j r) + p^4 r^3]^2, \quad (87)$$

$$g_i^{(+)} = F_i^2 \frac{r^4 f}{\ell^2} [b\ell r (3p^2 r + 4q_j) - q_j (2\ell^2 c^2 + q_j r) + p^4 r^3]^{-1}. \quad (88)$$

The reductions of the perturbation equations for $Y_{\pm i}$ to the master equations (71) are accomplished by means of the substitutions

$$Y_{+i} = V_i^{(\pm)} Z_i^{(\pm)} + (\Xi_i^{(\pm)} - 2i\varpi) \Lambda_- Z_i^{(\pm)} + \mathcal{F}_i^{(\pm)}, \quad (89)$$

$$Y_{-i} = V_i^{(\pm)} Z_i^{(\pm)} + (\Xi_i^{(\pm)} + 2i\varpi) \Lambda_+ Z_i^{(\pm)} + \mathcal{F}_i^{(\pm)}, \quad (90)$$

whose inverse transformations assume the form

$$K_i^{(\mp)} Z_i^{(\pm)} = \frac{r^8}{\mathcal{D}_i} Q_i Y_{+i} - \frac{r^8}{\mathcal{D}_i} (\Xi_i^{(\pm)} - 2i\varpi) \Lambda_+ Y_{+i} - \frac{r^8}{\mathcal{D}_i} [Q_i - (\Xi_i^{(\pm)} - 2i\varpi) \mathcal{L}_{-i}^{(\pm)}] \mathcal{F}_i^{(\pm)}, \quad (91)$$

$$K_i^{(\pm)} Z_i^{(\pm)} = \frac{r^8}{\mathcal{D}_i} Q_i Y_{-i} - \frac{r^8}{\mathcal{D}_i} (\Xi_i^{(\pm)} + 2i\varpi) \Lambda_- Y_{-i} - \frac{r^8}{\mathcal{D}_i} [Q_i - (\Xi_i^{(\pm)} + 2i\varpi) \mathcal{L}_{+i}^{(\pm)}] \mathcal{F}_i^{(\pm)}, \quad (92)$$

where $K_i^{(\pm)} = p^4 \pm 2i\varpi q_j$.

In the limit of $a \rightarrow 0$, parameters $\varpi = \gamma(\omega - am/\ell^2)$ and $p^2 = \gamma^2(m - a\omega)^2/\ell^2 + k^2$ tend, respectively, to the frequency and to the square of the perturbation wave vector. A lengthy but straightforward calculation shows that equations (71), with $\mathcal{F}_i^{(\pm)} = 0$ and potentials given by (77) and (78), reduce to the sourceless wave equations for the linear perturbations of a non-rotating charged black string [71] as $a \rightarrow 0$. The variables $Z_i^{(+)}$ ($i = 1, 2$) correspond to the even (polar) perturbations under the parity transformation $\varphi \rightarrow -\varphi$, while $Z_i^{(-)}$ ($i = 1, 2$) are associated to the odd (axial) sector of perturbations. In the zero charge limit, the gravitational and electromagnetic perturbations decouple, and the wave equations for $Z_1^{(+)}$ and $Z_1^{(-)}$ govern the electromagnetic perturbations of an electrically neutral black string, while those for $Z_2^{(+)}$ and $Z_2^{(-)}$ govern the gravitational perturbations. The source-free equations for $Z_1^{(+)}$ and $Z_1^{(-)}$ reduce, in the $Q \rightarrow 0$ limit, to the master equations for the Regge-Wheeler-Zerilli variables of the electromagnetic perturbations [72, 73].

5. The SUSY quantum mechanics of perturbations

In addition to allow the reduction of the problem to the four real Schrödinger-like wave equations (71), the transformation theory also reveals important aspects on the mathematical structure of the gravitoelectromagnetic perturbations. When properly taken into account, these aspects will simplify even more the study and application of the black-string perturbation theory.

A result which follows directly from the transformation theory is the existence of a relation, pair to pair, between the variables $Z_i^{(+)}$ of one sector of perturbations and the variables $Z_i^{(-)}$ of the other sector (see Appendix C). As we will show here, these relations can be viewed as a consequence of an underlying quantum-mechanical supersymmetry [74–76] of the black-string perturbation equations, a typical result for black holes in four spacetime dimensions [77–79].

To see the emergence of the supersymmetric (SUSY) aspects of the theory, we first notice that the potentials $V_i^{(\pm)}$ can be written as

$$V_i^{(\pm)} = \pm \frac{dW_i}{dr_*} + W_i^2 + \Omega_i^2, \quad (93)$$

with the introduction of the functions

$$W_i = \mathfrak{B}_i \mathfrak{f}_i + \frac{\mathfrak{Z}_i}{2\mathfrak{B}_i} = q_j \mathfrak{f}_i + i\Omega_i, \quad (94)$$

and the constants

$$\Omega_i = -i \frac{p^4}{2q_j}, \quad (i, j = 1, 2; i \neq j). \quad (95)$$

The Riccati form of equation (93) shows that $V_i^{(+)}$ and $V_i^{(-)}$ are supersymmetric partner potentials generated by the superpotentials W_i ($i = 1, 2$). In terms of W_i and Ω_i , the sourceless versions of equations (71) take the form

$$\left(-\frac{d^2}{dr_*^2} + W_i^2 \pm \frac{dW_i}{dr_*} \right) Z_i^{(\pm)} = (\varpi^2 - \Omega_i^2) Z_i^{(\pm)}, \quad (96)$$

which shows that $E_i = \varpi^2 - \Omega_i^2$ plays the role of the energy of the corresponding effective quantum-mechanical problem. From the SUSY quantum mechanics formalism, we know that equations (96) can be simplified with the introduction of the first-order operators

$$A_i = \frac{d}{dr_*} + W_i \quad \text{and} \quad A_i^\dagger = -\frac{d}{dr_*} + W_i. \quad (97)$$

The effective hamiltonians $H_i^{(\pm)}$ are then written as

$$H_i^{(-)} = A_i^\dagger A_i \quad \text{and} \quad H_i^{(+)} = A_i A_i^\dagger, \quad (98)$$

so that equations (96) assume the traditional form

$$H_i^{(\pm)} Z_i^{(\pm)} = E_i Z_i^{(\pm)}. \quad (99)$$

A consequence of the supersymmetric partnership between the effective potentials $V_i^{(+)}$ and $V_i^{(-)}$ is that wave functions of one sector of perturbations are interconnected to wave functions of the other sector. In fact, applying the operators $H_i^{(-)}$ and $H_i^{(+)}$, respectively, to the functions $A_i^\dagger Z_i^{(+)}$ and $A_i Z_i^{(-)}$, and using equations (99), we obtain

$$H_i^{(-)}(A_i^\dagger Z_i^{(+)}) = E_i(A_i^\dagger Z_i^{(+)}) \quad \text{and} \quad H_i^{(+)}(A_i Z_i^{(-)}) = E_i(A_i Z_i^{(-)}). \quad (100)$$

These relations imply that solutions for $Z_i^{(-)}$ and $Z_i^{(+)}$ with an eigenvalue E_i are proportional, respectively, to $A_i^\dagger Z_i^{(+)}$ and $A_i Z_i^{(-)}$; that is,

$$\mathcal{C}_i^{(+)} Z_i^{(-)} = A_i^\dagger Z_i^{(+)}, \quad (101)$$

and

$$\mathcal{C}_i^{(-)} Z_i^{(+)} = A_i Z_i^{(-)}, \quad (102)$$

where $\mathcal{C}_i^{(+)}$ and $\mathcal{C}_i^{(-)}$ are proportionality constants. Now applying A_i to both sides of (101), and using equations (99) and (102) to simplify the resulting expression, we find that the constants $\mathcal{C}_i^{(\pm)}$ are constrained by the following equation:

$$\mathcal{C}_i^{(+)} \mathcal{C}_i^{(-)} = E_i = \varpi^2 - \Omega_i^2. \quad (103)$$

Therefore, choosing the relative normalization of $Z_i^{(-)}$ and $Z_i^{(+)}$ such that

$$\mathcal{C}_i^{(+)} = \mathcal{C}_i^{(-)*} = i(\Omega_i + \varpi) = \frac{p^4}{2q_j} + i\varpi, \quad (104)$$

relations (101) and (102) become

$$(p^4 + 2i\varpi q_j) Z_i^{(-)} = \left(p^4 + \frac{2q_j^2}{F_i}\right) Z_i^{(+)} - 2q_j \frac{d}{dr_*} Z_i^{(+)}, \quad (105)$$

$$(p^4 - 2i\varpi q_j) Z_i^{(+)} = \left(p^4 + \frac{2q_j^2}{F_i}\right) Z_i^{(-)} + 2q_j \frac{d}{dr_*} Z_i^{(-)}, \quad (106)$$

which are identical to the relations (C.46) and (C.47) when we consider that \mathcal{A}_i and $\mathcal{F}_i^{(\pm)}$ vanish in the absence of sources, and $\kappa_i = p^4$ and $\beta_i = q_j$ for black-string perturbations.

6. Summary of results and perspectives

In this paper a set of four decoupled complex equations for the variables $Y_{\pm i}$ ($i = 1, 2$) was obtained by means of a gauge and tetrad invariant perturbation approach which includes the presence of sources. The variables $Y_{\pm i}$ are combinations of the spin coefficients, the Weyl and the Maxwell scalars of the Newman-Penrose formalism, and represent the radiative (non-trivial) parts of the gravitoelectromagnetic perturbations of rotating charged black strings.

With the aim of obtaining perturbation wave equations of a Schrödinger-like form, we have generalized in Appendix C the Chandrasekhar transformation theory [60–62] to deal with second-order ordinary differential equations with source terms. It is worth emphasizing here that the constructed transformation theory is general in character, and applies to any equation of the form (62) or (69) with $P_i = \partial_{r_*} \ln(r^8/\mathcal{D}_i)$ and arbitrary functions \mathcal{D}_i , Q_i and $\mathfrak{S}_{\pm i}$. An obvious application is to the study of perturbations of diverse black holes which admit a formulation in the manner of Teukolsky [68, 80]. In fact, given its high degree of generality, we can consider the development of this transformation theory as one of the main results of the present work.

The Chandrasekhar transformation theory with sources was then used to reduce the black-string perturbation problem to four real decoupled inhomogeneous Schrödinger-like equations for a new set of variables, $Z_i^{(-)}$ and $Z_i^{(+)}$. As a consequence of the way the rotation is implemented in the charged black strings, the main difference of the resulting wave equations in comparison with the non-rotating case is the Lorentz transformation of frequency and wave-vector components:

$$\omega \rightarrow \varpi = \gamma \left(\omega - \frac{am}{\ell^2} \right), \quad \left(\frac{m}{\ell} \right)^2 + k^2 \rightarrow p^2 = \frac{\gamma^2}{\ell^2} (m - a\omega)^2 + k^2. \quad (107)$$

The relation between the variables $Z_i^{(-)}$ and $Z_i^{(+)}$ reveals an underlying hidden symmetry of the four-dimensional Einstein-Maxwell theory on a black-hole spacetime. Such a symmetry can be viewed both as an extension of the electric/magnetic duality [81–85] and of its counterpart in linearized gravity [86–89]. When properly taken into account, this result may have important consequences in connection to the AdS/CFT correspondence, and may also be explored as a tool in future applications. For instance, it is well known that polar perturbation equations of static black holes are much more difficult to be solved (either analytically or numerically) than the axial wave equations. However, in view of the quantum-mechanical supersymmetry of the master equations, we can find solutions in the sector labeled by $(-)$ and use relation (106) to obtain the wave functions in the $(+)$ sector of perturbations.

Other aspect of the black-string perturbation theory, which is relevant for future applications, is the invariance of the set of variables $\{\Phi_0, \Phi_1, \mathcal{K}, \mathcal{S}\}$ and $\{\Phi_3, \Phi_4, \mathcal{N}, \mathcal{L}\}$ under infinitesimal tetrad transformations. As shown by Wald [90, 91], a tetrad-invariant approach for perturbations is an essential ingredient to allow the use of the Chrzanowski-Cohen-Kegeles procedure [92, 93] to reconstruct the metric and vector potentials from the perturbed NP scalars. Additionally, the explicit presence of source terms in the wave equations opens the possibility of a series of important applications, including the study of the influence of a hypothetical astrophysical environment on a rotating charged black string.

Appendix A. The Newman-Penrose formalism

The method of spin coefficients of Newman and Penrose [63] is a tetrad formalism with a basis of null vectors $\{\vec{l}, \vec{n}, \vec{m}, \vec{m}^*\}$, where \vec{l} and \vec{n} are real and \vec{m} and \vec{m}^* are complex

conjugates of each other. In order to introduce the notation and sign conventions used in this work and to make the paper self-contained, we present below the main equations of the Newman-Penrose formalism with the explicit inclusion of source terms due to an electromagnetic field and other nongravitational fields.

Appendix A.1. The null basis and the spin coefficients

The tetrad system of vectors $\{\vec{l}, \vec{n}, \vec{m}, \vec{m}^*\}$ are required to satisfy the orthogonality conditions,

$$l_\mu m^\mu = l_\mu m^{*\mu} = n_\mu m^\mu = n_\mu m^{*\mu} = 0, \quad (\text{A.1})$$

and the normalization conditions,

$$l_\mu n^\mu = -m_\mu m^{*\mu} = -1, \quad (\text{A.2})$$

besides, of course, the null-vector conditions,

$$l_\mu l^\mu = n_\mu n^\mu = m_\mu m^\mu = m_\mu^* m^{*\mu} = 0. \quad (\text{A.3})$$

As a result of these conditions, the metric takes the form of a flat-space metric in a null basis,

$$[\eta_{ab}] = [\eta^{ab}] = \begin{bmatrix} 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}, \quad (\text{A.4})$$

with the identifications $\vec{e}_1 = \vec{l}$, $\vec{e}_2 = \vec{n}$, $\vec{e}_3 = \vec{m}$ and $\vec{e}_4 = \vec{m}^*$.

The connection coefficients of the usual tensor calculus are substituted by 12 complex functions, which are called the spin coefficients and designated by the symbols

$$\begin{aligned} -\kappa &= \gamma_{ml}l = l_{\mu;\nu}m^\mu l^\nu, & -\rho &= \gamma_{mlm^*} = l_{\mu;\nu}m^\mu m^{*\nu}, \\ -\sigma &= \gamma_{mlm} = l_{\mu;\nu}m^\mu m^\nu, & -\mu &= \gamma_{nm^*m} = m_{\mu;\nu}^*n^\mu m^\nu, \\ -\lambda &= \gamma_{nm^*m^*} = m_{\mu;\nu}^*n^\mu m^{*\nu}, & -\tau &= \gamma_{mln} = l_{\mu;\nu}m^\mu n^\nu, \\ -\nu &= \gamma_{nm^*n} = m_{\mu;\nu}^*n^\mu n^\nu, & -\pi &= \gamma_{nm^*l} = m_{\mu;\nu}^*n^\mu l^\nu, \\ -\varepsilon &= \frac{1}{2}[\gamma_{nl}l + \gamma_{mm^*}l] = \frac{1}{2}[l_{\mu;\nu}n^\mu l^\nu + m_{\mu;\nu}^*m^\mu l^\nu], \\ -\gamma &= \frac{1}{2}[\gamma_{nl}n + \gamma_{mm^*}n] = \frac{1}{2}[l_{\mu;\nu}n^\mu n^\nu + m_{\mu;\nu}^*m^\mu n^\nu], \\ -\alpha &= \frac{1}{2}[\gamma_{nlm^*} + \gamma_{mm^*m^*}] = \frac{1}{2}[l_{\mu;\nu}n^\mu m^{*\nu} + m_{\mu;\nu}^*m^\mu m^{*\nu}], \\ -\beta &= \frac{1}{2}[\gamma_{nlm} + \gamma_{mm^*m}] = \frac{1}{2}[l_{\mu;\nu}n^\mu m^\nu + m_{\mu;\nu}^*m^\mu m^\nu], \end{aligned} \quad (\text{A.5})$$

where the quantities $\gamma_{abc} = e_a{}^\mu e_{b\mu;\nu} e_c{}^\nu$ are the so-called Ricci rotation coefficients of the general tetrad formalism. The basis vectors, when acting as directional derivatives, are also named by special symbols:

$$D = l^\mu \frac{\partial}{\partial x^\mu}, \quad \Delta = n^\mu \frac{\partial}{\partial x^\mu}, \quad \delta = m^\mu \frac{\partial}{\partial x^\mu}, \quad \delta^* = m^{*\mu} \frac{\partial}{\partial x^\mu}. \quad (\text{A.6})$$

These operators do not commute with each other and the Lie brackets among them give rise to the following commutation relations,

$$[\Delta, D] = \Delta D - D\Delta = (\gamma + \gamma^*)D + (\varepsilon + \varepsilon^*)\Delta - (\tau^* + \pi)\delta - (\tau + \pi^*)\delta^*, \quad (\text{A.7})$$

$$[\delta, D] = \delta D - D\delta = (\alpha^* + \beta - \pi^*)D + \kappa\Delta - (\rho^* + \varepsilon - \varepsilon^*)\delta - \sigma\delta^*, \quad (\text{A.8})$$

$$[\delta, \Delta] = \delta\Delta - \Delta\delta = -\nu^*D + (\tau - \alpha^* - \beta)\Delta + (\mu - \gamma + \gamma^*)\delta + \lambda^*\delta^*, \quad (\text{A.9})$$

$$[\delta^*, \delta] = \delta^*\delta - \delta\delta^* = (\mu^* - \mu)D + (\rho^* - \rho)\Delta + (\alpha - \beta^*)\delta + (\beta - \alpha^*)\delta^*, \quad (\text{A.10})$$

which are part of the basic set of equations of the NP formalism.

Appendix A.2. Weyl, Ricci and Maxwell scalars

In a Riemannian four-dimensional manifold, half of the independent components of the Riemann curvature tensor are given by the Ricci tensor $R_{ac} = \eta^{ad}R_{abcd}$, and the other half are given by the Weyl tensor

$$\begin{aligned} C_{abcd} = & R_{abcd} - \frac{1}{2} [\eta_{ac}R_{bd} + \eta_{bd}R_{ac} - \eta_{bc}R_{ad} - \eta_{ad}R_{bc}] \\ & + \frac{1}{6} [\eta_{ac}\eta_{bd} - \eta_{ad}\eta_{bc}] R, \end{aligned} \quad (\text{A.11})$$

where $R = \eta^{ab}R_{ab} = -2[R_{ln} - R_{mm^*}]$ is the curvature (Ricci) scalar.

The ten independent components of the Weyl tensor in a null basis are completely specified by the five complex scalars

$$\begin{aligned} \Psi_0 &= C_{lm lm} = C_{\mu\nu\rho\sigma} l^\mu m^\nu l^\rho m^\sigma, \\ \Psi_1 &= C_{ln lm} = C_{\mu\nu\rho\sigma} l^\mu n^\nu l^\rho m^\sigma, \\ \Psi_2 &= C_{lmm^*n} = C_{\mu\nu\rho\sigma} l^\mu m^\nu m^{*\rho} n^\sigma, \\ \Psi_3 &= C_{lnm^*n} = C_{\mu\nu\rho\sigma} l^\mu n^\nu m^{*\rho} n^\sigma, \\ \Psi_4 &= C_{nmm^*nm^*} = C_{\mu\nu\rho\sigma} n^\mu m^{*\nu} n^\rho m^{*\sigma}, \end{aligned} \quad (\text{A.12})$$

and the components of the Ricci tensor are represented by the quantities

$$\begin{aligned} \Phi_{00} &= \frac{1}{2}R_{ll} = \frac{1}{2}R_{\mu\nu}l^\mu l^\nu, & \Phi_{22} &= \frac{1}{2}R_{nn} = \frac{1}{2}R_{\mu\nu}n^\mu n^\nu, \\ \Phi_{01} &= \frac{1}{2}R_{lm} = \frac{1}{2}R_{\mu\nu}l^\mu m^\nu, & \Phi_{10} &= \frac{1}{2}R_{lm^*} = \frac{1}{2}R_{\mu\nu}l^\mu m^{*\nu}, \\ \Phi_{02} &= \frac{1}{2}R_{mm} = \frac{1}{2}R_{\mu\nu}m^\mu m^\nu, & \Phi_{20} &= \frac{1}{2}R_{m^*m^*} = \frac{1}{2}R_{\mu\nu}m^{*\mu}m^{*\nu}, \\ \Phi_{12} &= \frac{1}{2}R_{nm} = \frac{1}{2}R_{\mu\nu}n^\mu m^\nu, & \Phi_{21} &= \frac{1}{2}R_{nm^*} = \frac{1}{2}R_{\mu\nu}n^\mu m^{*\nu}, \\ \Phi_{11} &= \frac{1}{4}[R_{ln} + R_{mm^*}] = \frac{1}{4}[R_{\mu\nu}l^\mu n^\nu + R_{\mu\nu}m^\mu m^{*\nu}], \\ \Pi &= \frac{1}{24}R = -\frac{1}{12}[R_{ln} - R_{mm^*}] = -\frac{1}{12}[R_{\mu\nu}l^\mu n^\nu - R_{\mu\nu}m^\mu m^{*\nu}]. \end{aligned} \quad (\text{A.13})$$

On basis of equations (A.11)-(A.13) and the symmetry properties of curvature tensors, one finds that the tetrad components of the Riemman tensor are related to the Weyl

and Ricci scalars by

$$\begin{aligned}
R_{lnmm^*} &= -\Psi_2 + \Psi_2^*, & R_{lmml} &= \Psi_0, & R_{lmnm^*} &= -\Psi_2 + 2\Pi, & R_{nmnm} &= \Psi_4^*, \\
R_{lnln} &= \Psi_2 + \Psi_2^* + 2\Phi_{11} - 2\Pi, & R_{lnlm} &= \Psi_1 + \Phi_{01}, & R_{lmmlm^*} &= \Phi_{00}, \\
R_{mm^*mm^*} &= \Psi_2 + \Psi_2^* - 2\Phi_{11} - 2\Pi, & R_{lmnm^*} &= -\Psi_1 + \Phi_{01}, & R_{nmnm^*} &= \Phi_{22}, \\
R_{lnnm} &= -\Psi_3^* - \Phi_{12}, & R_{nmnm^*} &= -\Psi_3^* + \Phi_{12}, & R_{mlmn} &= \Phi_{02}.
\end{aligned} \tag{A.14}$$

In a spacetime with vanishing cosmological constant, the Ricci tensor is identically zero in the vacuum and the spacetime curvature is given only by the Weyl scalars. In the general case, however, the Ricci scalars must be considered and their values are fixed by the Einstein field equations,

$$R_{ab} = 8\pi G \left[T_{ab} - \frac{1}{2} T \eta_{ab} \right] + \Lambda_c \eta_{ab}, \tag{A.15}$$

where Λ_c is the cosmological constant and T_{ab} is the energy-momentum tensor of all non-gravitational fields (including matter). In the present work, the cosmological constant takes the value $\Lambda_c = -3/\ell^2$ and the tensor T_{ab} is conveniently separated into an electromagnetic component,

$$T_{ab}^{(\text{EM})} = \frac{\ell^2}{4\pi G} \left[F^c{}_a F_{cb} - \frac{1}{4} \eta_{ab} F_{cd} F^{cd} \right], \tag{A.16}$$

and an energy-momentum tensor $T_{ab}^{(\text{MAT})}$ for the matter and the remaining fields.

The six non-vanishing components of the Maxwell tensor $F_{\mu\nu}$ are replaced, in the Newman-Penrose formalism, by three complex scalars defined as follows

$$\begin{aligned}
\phi_0 &= F_{lm} = F_{\mu\nu} l^\mu m^\nu, \\
\phi_1 &= \frac{1}{2} [F_{ln} + F_{m^*m}] = \frac{1}{2} F_{\mu\nu} [l^\mu n^\nu + m^{*\mu} m^\nu], \\
\phi_2 &= F_{m^*n} = F_{\mu\nu} m^{*\mu} n^\nu.
\end{aligned} \tag{A.17}$$

Using equations (A.15)-(A.17), we can write the Ricci scalars (A.13) in terms of the Maxwell scalars and the tetrad components of the tensor $T_{\mu\nu}^{(\text{MAT})}$:

$$\begin{aligned}
\Phi_{00} &= 2\ell^2 \phi_0 \phi_0^* + 4\pi G T_{ll}^{(\text{MAT})}; & \Phi_{22} &= 2\ell^2 \phi_2 \phi_2^* + 4\pi G T_{nn}^{(\text{MAT})}; \\
\Phi_{01} &= 2\ell^2 \phi_0 \phi_1^* + 4\pi G T_{lm}^{(\text{MAT})}; & \Phi_{10} &= 2\ell^2 \phi_1 \phi_0^* + 4\pi G T_{lm^*}^{(\text{MAT})}; \\
\Phi_{02} &= 2\ell^2 \phi_0 \phi_2^* + 4\pi G T_{mm}^{(\text{MAT})}; & \Phi_{20} &= 2\ell^2 \phi_2 \phi_0^* + 4\pi G T_{m^*m^*}^{(\text{MAT})}; \\
\Phi_{12} &= 2\ell^2 \phi_1 \phi_2^* + 4\pi G T_{nm}^{(\text{MAT})}; & \Phi_{21} &= 2\ell^2 \phi_2 \phi_1^* + 4\pi G T_{nm^*}^{(\text{MAT})}; \\
\Phi_{11} &= 2\ell^2 \phi_1 \phi_1^* + 2\pi G [T_{ln}^{(\text{MAT})} + T_{mm^*}^{(\text{MAT})}]; & \Pi &= -\frac{1}{3} \pi G T^{(\text{MAT})} - \frac{1}{2} \ell^{-2}.
\end{aligned} \tag{A.18}$$

Appendix A.3. The Ricci equations

In the Newman-Penrose formalism, the Ricci identities

$$R_{abcd} = \gamma_{abd,c} - \gamma_{abc,d} + \gamma_{fac} \gamma_b^f{}_d - \gamma_{fad} \gamma_b^f{}_c + \gamma_{baf} \left(\gamma_c^f{}_d - \gamma_d^f{}_c \right) \tag{A.19}$$

comprise a set of 18 complex equations involving the spin coefficients and the Weyl and the Ricci scalars. For completeness, we present the Ricci equations below, following the notation and sign conventions adopted in this paper:

$$D\rho - \delta^*\kappa = (\rho^2 + \sigma\sigma^*) + \rho(\varepsilon + \varepsilon^*) - \kappa^*\tau - \kappa(3\alpha + \beta^* - \pi) + \Phi_{00}; \quad (\text{a})$$

$$D\sigma - \delta\kappa = \sigma(3\varepsilon - \varepsilon^* + \rho + \rho^*) + \kappa(\pi^* - \tau - 3\beta - \alpha^*) + \Psi_0; \quad (\text{b})$$

$$D\tau - \Delta\kappa = \rho(\tau + \pi^*) + \sigma(\tau^* + \pi) + \tau(\varepsilon - \varepsilon^*) - \kappa(3\gamma + \gamma^*) + \Psi_1 + \Phi_{01}; \quad (\text{c})$$

$$D\alpha - \delta^*\varepsilon = \alpha(\rho + \varepsilon^* - 2\varepsilon) + \beta\sigma^* - \beta^*\varepsilon - \kappa\lambda - \kappa^*\gamma + \pi(\varepsilon + \rho) + \Phi_{10}; \quad (\text{d})$$

$$D\beta - \delta\varepsilon = \sigma(\alpha + \pi) + \beta(\rho^* - \varepsilon^*) - \kappa(\mu + \gamma) - \varepsilon(\alpha^* - \pi^*) + \Psi_1; \quad (\text{e})$$

$$D\gamma - \Delta\varepsilon = \alpha(\tau + \pi^*) + \beta(\tau^* + \pi) - \gamma(\varepsilon + \varepsilon^*) - \varepsilon(\gamma + \gamma^*) + \tau\pi - \nu\kappa + \Psi_2 + \Phi_{11} - \Pi; \quad (\text{f})$$

$$D\lambda - \delta^*\pi = (\rho\lambda + \sigma^*\mu) + \pi(\pi + \alpha - \beta) - \nu\kappa^* - \lambda(3\varepsilon - \varepsilon^*) + \Phi_{20}; \quad (\text{g})$$

$$D\mu - \delta\pi = (\rho^*\mu + \sigma\lambda) + \pi(\pi^* - \alpha^* + \beta) - \mu(\varepsilon + \varepsilon^*) - \nu\kappa + \Psi_2 + 2\Pi; \quad (\text{h})$$

$$D\nu - \Delta\pi = \mu(\pi + \tau^*) + \lambda(\pi^* + \tau) + \pi(\gamma - \gamma^*) - \nu(3\varepsilon + \varepsilon^*) + \Psi_3 + \Phi_{21}; \quad (\text{i})$$

$$\Delta\lambda - \delta^*\nu = -\lambda(\mu + \mu^* + 3\gamma - \gamma^*) + \nu(3\alpha + \beta^* + \pi - \tau^*) - \Psi_4; \quad (\text{j})$$

$$\delta\rho - \delta^*\sigma = \rho(\alpha^* + \beta) - \sigma(3\alpha - \beta^*) + \tau(\rho - \rho^*) + \kappa(\mu - \mu^*) - \Psi_1 + \Phi_{01}; \quad (\text{k})$$

$$\delta\alpha - \delta^*\beta = (\mu\rho - \lambda\sigma) + \alpha\alpha^* + \beta\beta^* - 2\alpha\beta + \gamma(\rho - \rho^*) + \varepsilon(\mu - \mu^*) - \Psi_2 + \Phi_{11} + \Pi; \quad (\text{l})$$

$$\delta\lambda - \delta^*\mu = \nu(\rho - \rho^*) + \pi(\mu - \mu^*) + \mu(\alpha + \beta^*) + \lambda(\alpha^* - 3\beta) - \Psi_3 + \Phi_{21}; \quad (\text{m})$$

$$\delta\nu - \Delta\mu = (\mu^2 + \lambda\lambda^*) + \mu(\gamma + \gamma^*) - \nu^*\pi + \nu(\tau - 3\beta - \alpha^*) + \Phi_{22}; \quad (\text{n})$$

$$\delta\gamma - \Delta\beta = \gamma(\tau - \alpha^* - \beta) + \mu\tau - \sigma\nu - \varepsilon\nu^* - \beta(\gamma - \gamma^* - \mu) + \alpha\lambda^* + \Phi_{12}; \quad (\text{o})$$

$$\begin{aligned}\delta\tau - \Delta\sigma &= (\mu\sigma + \lambda^*\rho) + \tau(\tau + \beta - \alpha^*) \\ &\quad - \sigma(3\gamma - \gamma^*) - \kappa\nu^* + \Phi_{02};\end{aligned}\tag{p}$$

$$\begin{aligned}\Delta\rho - \delta^*\tau &= -(\rho\mu^* + \sigma\lambda) + \tau(\beta^* - \alpha - \tau^*) \\ &\quad + \rho(\gamma + \gamma^*) + \nu\kappa - \Psi_2 - 2\Pi;\end{aligned}\tag{q}$$

$$\begin{aligned}\Delta\alpha - \delta^*\gamma &= \nu(\rho + \varepsilon) - \lambda(\tau + \beta) + \alpha(\gamma^* - \mu^*) \\ &\quad + \gamma(\beta^* - \tau^*) - \Psi_3.\end{aligned}\tag{r}$$

(A.20)

From the complete set of Ricci equations, we are mainly interested here in the equations (A.20b), (A.20c), (A.20i), (A.20j), (A.20k), (A.20m), which are important in the study of perturbations of rotating charged black strings. By using relations (A.18) it is possible to rewrite such equations in the following form:

$$D\sigma - \delta\kappa = \sigma(3\varepsilon - \varepsilon^* + \rho + \rho^*) + \kappa(\pi^* - \tau - 3\beta - \alpha^*) + \Psi_0;\tag{A.21}$$

$$\begin{aligned}D\tau - \Delta\kappa &= \rho(\tau + \pi^*) + \sigma(\tau^* + \pi) + \tau(\varepsilon - \varepsilon^*) \\ &\quad - \kappa(3\gamma + \gamma^*) + \Psi_1 + 2\ell^2\phi_0\phi_1^* + 4\pi G T_{lm}^{(\text{MAT})};\end{aligned}\tag{A.22}$$

$$\begin{aligned}D\nu - \Delta\pi &= \mu(\pi + \tau^*) + \lambda(\pi^* + \tau) + \pi(\gamma - \gamma^*) \\ &\quad - \nu(3\varepsilon + \varepsilon^*) + \Psi_3 + 2\ell^2\phi_2\phi_1^* + 4\pi G T_{nm}^{(\text{MAT})};\end{aligned}\tag{A.23}$$

$$\Delta\lambda - \delta^*\nu = -\lambda(\mu + \mu^* + 3\gamma - \gamma^*) + \nu(3\alpha + \beta^* + \pi - \tau^*) - \Psi_4;\tag{A.24}$$

$$\begin{aligned}\delta\rho - \delta^*\sigma &= \rho(\alpha^* + \beta) - \sigma(3\alpha - \beta^*) + \tau(\rho - \rho^*) \\ &\quad + \kappa(\mu - \mu^*) - \Psi_1 + 2\ell^2\phi_0\phi_1^* + 4\pi G T_{lm}^{(\text{MAT})};\end{aligned}\tag{A.25}$$

$$\begin{aligned}\delta\lambda - \delta^*\mu &= \nu(\rho - \rho^*) + \pi(\mu - \mu^*) + \mu(\alpha + \beta^*) \\ &\quad + \lambda(\alpha^* - 3\beta) - \Psi_3 + 2\ell^2\phi_2\phi_1^* + 4\pi G T_{nm}^{(\text{MAT})}.\end{aligned}\tag{A.26}$$

Appendix A.4. The Bianchi identities

The twenty linearly independent Bianchi identities are given by eight complex equations,

$$\begin{aligned}R_{lm[lm|m^*]} &= 0, & R_{lm[nl|m^*]} &= 0, & R_{lm[lm|n]} &= 0, & R_{lm[m^*m|n]} &= 0, \\ R_{m^*n[lm|m^*]} &= 0, & R_{m^*n[nl|m^*]} &= 0, & R_{m^*n[lm|n]} &= 0, & R_{m^*n[m^*m|n]} &= 0,\end{aligned}\tag{A.27}$$

and by four real equations that are derived from

$$\eta^{bc}(R_{ab} - \tfrac{1}{2}\eta_{ab}R)_{|c} = 0,\tag{A.28}$$

where above we have used square brackets to denote anti-symmetrization and a vertical bar to indicate the intrinsic derivative, $\Omega_{a_1\dots a_n|b} = \Omega_{\mu_1\dots\mu_n;\nu} e_{a_1}^{\mu_1} \dots e_{a_n}^{\mu_n} e_b^{\nu}$.

In terms of the Newman-Penrose quantities, the identities (A.27) become

$$\begin{aligned}
& (\delta^* - 4\alpha + \pi)\Psi_0 - (D - 4\rho - 2\varepsilon)\Psi_1 - 3\kappa\Psi_2 = -(D - 2\varepsilon - 2\rho^*)\Phi_{01} \\
& \quad + (\delta + \pi^* - 2\alpha^* - 2\beta)\Phi_{00} + 2\sigma\Phi_{10} - 2\kappa\Phi_{11} - \kappa^*\Phi_{02}, \quad (a) \\
& (\delta^* + 2\pi - 2\alpha)\Psi_1 - (D - 3\rho)\Psi_2 - \lambda\Psi_0 - 2\kappa\Psi_3 = -(\delta^* - 2\alpha - 2\tau^*)\Phi_{01} \\
& \quad + (\Delta + \mu^* - 2\gamma - 2\gamma^*)\Phi_{00} - 2\rho\Phi_{11} - \sigma^*\Phi_{02} + 2\tau\Phi_{10} + 2D\Pi, \quad (b) \\
& (\delta^* + 3\pi)\Psi_2 - (D + 2\varepsilon - 2\rho)\Psi_3 - 2\lambda\Psi_1 - \kappa\Psi_4 = -(D - 2\rho^* + 2\varepsilon)\Phi_{21} \\
& \quad + (\delta - 2\alpha^* + 2\beta + \pi^*)\Phi_{20} - 2\mu\Phi_{10} + 2\pi\Phi_{11} - \kappa^*\Phi_{22} - 2\delta^*\Pi, \quad (c) \\
& (\delta^* + 4\pi + 2\alpha)\Psi_3 - (D + 4\varepsilon - \rho)\Psi_4 - 3\lambda\Psi_2 = (\Delta + \mu^* + 2\gamma - 2\gamma^*)\Phi_{20} \\
& \quad - (\delta^* + 2\alpha - 2\tau^*)\Phi_{21} - 2\nu\Phi_{10} - \sigma^*\Phi_{22} + 2\lambda\Phi_{11}, \quad (d) \\
& (\Delta - 4\gamma + \mu)\Psi_0 - (\delta - 4\tau - 2\beta)\Psi_1 - 3\sigma\Psi_2 = -(D - \rho^* - 2\varepsilon + 2\varepsilon^*)\Phi_{02} \\
& \quad + (\delta + 2\pi^* - 2\beta)\Phi_{01} - 2\kappa\Phi_{12} - \lambda^*\Phi_{00} + 2\sigma\Phi_{11}, \quad (e) \\
& (\Delta - 2\gamma + 2\mu)\Psi_1 - (\delta - 3\tau)\Psi_2 - \nu\Psi_0 - 2\sigma\Psi_3 = (\Delta + 2\mu^* - 2\gamma)\Phi_{01} \\
& \quad - (\delta^* - \tau^* + 2\beta^* - 2\alpha)\Phi_{02} - 2\rho\Phi_{12} - \nu^*\Phi_{00} + 2\tau\Phi_{11} + 2\delta\Pi, \quad (f) \\
& (\Delta + 3\mu)\Psi_2 - (\delta + 2\beta - 2\tau)\Psi_3 - 2\nu\Psi_1 - \sigma\Psi_4 = -(D - \rho^* + 2\varepsilon + 2\varepsilon^*)\Phi_{22} \\
& \quad + (\delta + 2\pi^* + 2\beta)\Phi_{21} - 2\mu\Phi_{11} - \lambda^*\Phi_{20} + 2\pi\Phi_{12} - 2\Delta\Pi, \quad (g) \\
& (\Delta + 2\gamma + 4\mu)\Psi_3 - (\delta - \tau + 4\beta)\Psi_4 - 3\nu\Psi_2 = (\Delta + 2\mu^* + 2\gamma)\Phi_{21} \\
& \quad - (\delta^* - \tau^* + 2\alpha + 2\beta^*)\Phi_{22} - 2\nu\Phi_{11} - \nu^*\Phi_{20} + 2\lambda\Phi_{12}. \quad (h)
\end{aligned}
\tag{A.29}$$

The contracted Bianchi identities (A.28), in turn, may be written as

$$\begin{aligned}
& \delta^*\Phi_{01} + \delta\Phi_{10} - D(\Phi_{11} + 3\Pi) - \Delta\Phi_{00} = \kappa^*\Phi_{12} + \kappa\Phi_{21} + (2\alpha + 2\tau^* - \pi)\Phi_{01} \\
& \quad + (2\alpha^* + 2\tau - \pi^*)\Phi_{10} - 2(\rho + \rho^*)\Phi_{11} - \sigma^*\Phi_{02} - \sigma\Phi_{20} + (\mu + \mu^* - 2\gamma - 2\gamma^*)\Phi_{00}, \quad (i) \\
& \delta^*\Phi_{12} + \delta\Phi_{21} - \Delta(\Phi_{11} + 3\Pi) - D\Phi_{22} = -\nu\Phi_{01} - \nu^*\Phi_{10} + (\tau^* - 2\beta^* - 2\pi)\Phi_{12} \\
& \quad + (\tau - 2\beta - 2\pi^*)\Phi_{21} + 2(\mu + \mu^*)\Phi_{11} - (\rho + \rho^* - 2\varepsilon - 2\varepsilon^*)\Phi_{22} + \lambda\Phi_{02} + \lambda^*\Phi_{20}, \quad (j) \\
& \delta(\Phi_{11} - 3\Pi) - D\Phi_{12} - \Delta\Phi_{01} + \delta^*\Phi_{02} = \kappa\Phi_{22} - \nu^*\Phi_{00} + (\tau^* - \pi + 2\alpha - 2\beta^*)\Phi_{02} \\
& \quad - \sigma\Phi_{21} + \lambda^*\Phi_{10} + 2(\tau - \pi^*)\Phi_{11} - (2\rho + \rho^* - 2\varepsilon^*)\Phi_{12} + (2\mu^* + \mu - 2\gamma)\Phi_{01}. \quad (k)
\end{aligned}
\tag{A.30}$$

The commutation relations (A.7)-(A.10), the Ricci equations (A.20) and the Bianchi identities (A.29) and (A.30) constitute the basic set of equations of the NP formalism.

Appendix A.5. Maxwell equations and the Ricci terms

In spacetimes with electromagnetic fields, the NP equations are supplemented by the Maxwell equations,

$$F_{[\mu\nu;\rho]} = 0, \quad \eta^{\nu\rho}F_{\mu\nu;\rho} = 4\pi J_\mu. \tag{A.31}$$

Projecting these equations on the complex null vectors $\{\vec{l}, \vec{n}, \vec{m}, \vec{m}^*\}$ and combining the resulting equations in a suitable form, one gets

$$(\delta^* + \pi - 2\alpha)\phi_0 - (D - 2\rho)\phi_1 - \kappa\phi_2 = 2\pi J_l, \quad (\text{A.32})$$

$$(\Delta + 2\mu)\phi_1 - (\delta - \tau + 2\beta)\phi_2 - \nu\phi_0 = 2\pi J_n, \quad (\text{A.33})$$

$$(\Delta + \mu - 2\gamma)\phi_0 - (\delta - 2\tau)\phi_1 - \sigma\phi_2 = 2\pi J_m, \quad (\text{A.34})$$

$$(\delta^* + 2\pi)\phi_1 - (D - \rho + 2\varepsilon)\phi_2 - \lambda\phi_0 = 2\pi J_{m^*}. \quad (\text{A.35})$$

The electromagnetic contribution to the Ricci (source) terms, which appear on the right hand side of equations (A.29), can be simplified with the use of the Maxwell equations (A.32)-(A.35). For instance, considering the expressions (A.18) for the Ricci scalars and using (A.32) to eliminate $D\phi_1^* - \delta\phi_0^*$, we have

$$\begin{aligned} -D\Phi_{01} + \delta\Phi_{00} &= 2\ell^2 [-\phi_0(D\phi_1^* - \delta\phi_0^*) - \phi_1^*D\phi_0 + \phi_0^*\delta\phi_0] + \dots \\ &= 2\ell^2 [-\phi_0\{(\pi^* - 2\alpha^*)\phi_0^* + 2\rho^*\phi_1^* - \kappa^*\phi_2^* - 2\pi J_l\} \\ &\quad - \phi_1^*D\phi_0 + \phi_0^*\delta\phi_0] + \dots, \end{aligned} \quad (\text{A.36})$$

and substituting this expression in equation (A.29a), we obtain

$$2\ell^2 [-\phi_1^*D\phi_0 + \phi_0^*\delta\phi_0 + 2(\varepsilon\phi_0\phi_1^* + \sigma\phi_1\phi_0^* - \kappa\phi_1\phi_1^* - \beta\phi_0\phi_0^*)] + 4\pi\ell^2\phi_0J_l + \dots, \quad (\text{A.37})$$

where the ellipses stand for matter, cosmological constant and other non-electromagnetic field contributions to the Ricci term in equation (A.29a).

With a similar sequence of replacements and simplifications, we find that the Ricci terms of equations (A.29) for a spacetime with electromagnetic field take the form

$$\begin{aligned} 2\ell^2 [-\phi_1^*D\phi_0 + \phi_0^*\delta\phi_0 + 2(\varepsilon\phi_0\phi_1^* + \sigma\phi_1\phi_0^* - \kappa\phi_1\phi_1^* - \beta\phi_0\phi_0^*)] + 4\pi\ell^2\phi_0J_l + \dots, & \quad (\text{a}) \\ 2\ell^2 [-\phi_1^*\delta^*\phi_0 + \phi_0^*\Delta\phi_0 + 2(\alpha\phi_0\phi_1^* - \rho\phi_1\phi_1^* - \gamma\phi_0\phi_0^* + \tau\phi_1\phi_0^*)] + 4\pi\ell^2\phi_0J_{m^*} + \dots, & \quad (\text{b}) \\ 2\ell^2 [-\phi_1^*D\phi_2 + \phi_0^*\delta\phi_2 - 2(\varepsilon\phi_2\phi_1^* + \mu\phi_1\phi_0^* - \beta\phi_2\phi_0^* - \pi\phi_1\phi_1^*)] + 4\pi\ell^2\phi_2J_l + \dots, & \quad (\text{c}) \\ 2\ell^2 [-\phi_1^*\delta^*\phi_2 + \phi_0^*\Delta\phi_2 - 2(\alpha\phi_2\phi_1^* + \nu\phi_1\phi_0^* - \gamma\phi_2\phi_0^* - \lambda\phi_1\phi_1^*)] + 4\pi\ell^2\phi_2J_{m^*} + \dots, & \quad (\text{d}) \\ 2\ell^2 [-\phi_2^*D\phi_0 + \phi_1^*\delta\phi_0 - 2(\kappa\phi_1\phi_2^* + \beta\phi_0\phi_1^* - \sigma\phi_1\phi_1^* - \varepsilon\phi_0\phi_2^*)] + 4\pi\ell^2\phi_0J_m + \dots, & \quad (\text{e}) \\ 2\ell^2 [-\phi_2^*\delta^*\phi_0 + \phi_1^*\Delta\phi_0 - 2(\rho\phi_1\phi_2^* + \gamma\phi_0\phi_1^* - \tau\phi_1\phi_1^* - \alpha\phi_0\phi_2^*)] + 4\pi\ell^2\phi_0J_n + \dots, & \quad (\text{f}) \\ 2\ell^2 [-\phi_2^*D\phi_2 + \phi_1^*\delta\phi_2 - 2(\varepsilon\phi_2\phi_2^* + \mu\phi_1\phi_1^* - \beta\phi_2\phi_1^* - \pi\phi_1\phi_2^*)] + 4\pi\ell^2\phi_2J_m + \dots, & \quad (\text{g}) \\ 2\ell^2 [-\phi_2^*\delta^*\phi_2 + \phi_1^*\Delta\phi_2 - 2(\alpha\phi_2\phi_2^* + \nu\phi_1\phi_1^* - \gamma\phi_2\phi_1^* - \lambda\phi_1\phi_2^*)] + 4\pi\ell^2\phi_2J_n + \dots. & \quad (\text{h}) \end{aligned} \quad (\text{A.38})$$

The Bianchi identities (A.29a), (A.29d), (A.29e), (A.29h) are particularly important for the study of the perturbations of rotating charged black strings developed in the present work. In view of that, we take into account the foregoing Ricci terms and rewrite these equations in full form:

$$\begin{aligned} (\delta^* - 4\alpha + \pi)\Psi_0 - (D - 4\rho - 2\varepsilon)\Psi_1 - 3\kappa\Psi_2 &= 2\ell^2 [-\phi_1^*D\phi_0 + \phi_0^*\delta\phi_0 + 2(\varepsilon\phi_0\phi_1^* \\ &\quad + \sigma\phi_1\phi_0^* - \kappa\phi_1\phi_1^* - \beta\phi_0\phi_0^*)] + 4\pi\ell^2\phi_0J_l + 4\pi G [-(D - 2\varepsilon - 2\rho^*)T_{lm}^{(\text{MAT})} \\ &\quad + (\delta + \pi^* - 2\alpha^* - 2\beta)T_{ll}^{(\text{MAT})} + 2\sigma T_{lm^*}^{(\text{MAT})} - \kappa(T_{ln}^{(\text{MAT})} + T_{mm^*}^{(\text{MAT})}) - \kappa^*T_{mm}^{(\text{MAT})}] ; \end{aligned} \quad (\text{A.39})$$

$$\begin{aligned}
(\delta^* + 4\pi + 2\alpha)\Psi_3 - (D + 4\varepsilon - \rho)\Psi_4 - 3\lambda\Psi_2 = 2\ell^2 [-\phi_1^*\delta^*\phi_2 + \phi_0^*\Delta\phi_2 - 2(\alpha\phi_2\phi_1^* \\
+ \nu\phi_1\phi_0^* - \gamma\phi_2\phi_0^* - \lambda\phi_1\phi_1^*)] + 4\pi\ell^2\phi_2J_{m^*} + 4\pi G [(\Delta + \mu^* + 2\gamma - 2\gamma^*)T_{m^*m^*}^{(\text{MAT})} \\
- (\delta^* + 2\alpha - 2\tau^*)T_{nm^*}^{(\text{MAT})} - 2\nu T_{lm^*}^{(\text{MAT})} - \sigma^*T_{nn}^{(\text{MAT})} + \lambda(T_{ln}^{(\text{MAT})} + T_{mm^*}^{(\text{MAT})})];
\end{aligned} \tag{A.40}$$

$$\begin{aligned}
(\Delta - 4\gamma + \mu)\Psi_0 - (\delta - 4\tau - 2\beta)\Psi_1 - 3\sigma\Psi_2 = 2\ell^2 [-\phi_2^*D\phi_0 + \phi_1^*\delta\phi_0 - 2(\kappa\phi_1\phi_2^* \\
+ \beta\phi_0\phi_1^* - \sigma\phi_1\phi_1^* - \varepsilon\phi_0\phi_2^*)] + 4\pi\ell^2\phi_0J_m + 4\pi G [-(D - \rho^* - 2\varepsilon + 2\varepsilon^*)T_{mm}^{(\text{MAT})} \\
+ (\delta + 2\pi^* - 2\beta)T_{lm}^{(\text{MAT})} - 2\kappa T_{nm}^{(\text{MAT})} - \lambda^*T_{ll}^{(\text{MAT})} + \sigma(T_{ln}^{(\text{MAT})} + T_{mm^*}^{(\text{MAT})})];
\end{aligned} \tag{A.41}$$

$$\begin{aligned}
(\Delta + 2\gamma + 4\mu)\Psi_3 - (\delta - \tau + 4\beta)\Psi_4 - 3\nu\Psi_2 = 2\ell^2 [-\phi_2^*\delta^*\phi_2 + \phi_1^*\Delta\phi_2 - 2(\alpha\phi_2\phi_2^* \\
+ \nu\phi_1\phi_1^* - \gamma\phi_2\phi_1^* - \lambda\phi_1\phi_2^*)] + 4\pi\ell^2\phi_2J_n + 4\pi G [(\Delta + 2\mu^* + 2\gamma)T_{nm^*}^{(\text{MAT})} \\
- (\delta^* - \tau^* + 2\alpha + 2\beta^*)T_{nn}^{(\text{MAT})} - \nu(T_{ln}^{(\text{MAT})} + T_{mm^*}^{(\text{MAT})}) - \nu^*T_{m^*m^*}^{(\text{MAT})} + 2\lambda T_{nm}^{(\text{MAT})}].
\end{aligned} \tag{A.42}$$

Appendix B. Gauge invariance and tetrad transformations

The physical quantities in the NP formalism are obtained from the projection of tensor fields on a basis of null vectors. As a result, the equations of the formalism involve only scalar functions, and so these quantities are independent of the choice of the coordinate system or, equivalently, they are invariant under *gauge transformations of the first kind*, as defined by Sachs [69]. In a general-relativistic perturbation theory, there is also the gauge freedom associated to the choice of the point identification map between the physical spacetime and the unperturbed (background) spacetime [94]. The so-called *gauge transformation of the second kind* is a change of identification map that can be represented by

$$x_{\text{new}}^\mu = x_{\text{old}}^\mu + \xi^\mu, \tag{B.1}$$

where ξ^μ is an infinitesimal arbitrary vector field. As a result of this transformation, a certain NP quantity T change as

$$T^{\text{new}} = T^{\text{old}} - T_{,\mu}^{\text{old}} \xi^\mu, \tag{B.2}$$

or, considering the first-order perturbation $T^{(1)}$ of the scalar T , we have

$$T^{(1)\text{new}} = T^{(1)\text{old}} - T_{,\mu}^{\text{old}} \xi^\mu. \tag{B.3}$$

In the case of a rotating charged black string in the Kinnersley frame (6)-(8), the only non-vanishing Weyl and Maxwell scalars are respectively Ψ_2 and ϕ_1 , and the only non-vanishing spin coefficients are ρ , μ and γ . According to the transformation (B.3), these quantities with non-vanishing background values depend on the point identification map, while the remaining NP scalars are invariant under the gauge transformations

(B.1). However, once we have chosen a specific null basis, it is also possible to perform a Lorentz transformation on this tetrad so that the metric (A.4) remains unchanged. Then, another relevant question to the study of linear perturbations in the Newman-Penrose formalism is what kind of quantities are invariant under infinitesimal changes in the tetrad.

Associated to the six parameters of the Lorentz group of transformations, there are six degrees of freedom related to rotations on the specific chosen null basis. For the analysis of the changes induced by these rotations on the various NP quantities, it is convenient to separate the rotations in three classes, based on their effect on the vectors $\{\vec{l}, \vec{n}, \vec{m}, \vec{m}^*\}$: (a) null rotations which leave \vec{l} unchanged are of class I; (b) null rotations which leave \vec{n} unchanged are of class II; and (c) transformations of class III consist of boosts and rotations which leave the directions of \vec{l} and \vec{n} unchanged, while rotate \vec{m} and \vec{m}^* by an angle θ . These rotations induce the following changes in the null tetrad basis:

$$\begin{aligned} \text{I: } & \vec{l} \rightarrow \vec{l}, \quad \vec{m} \rightarrow \vec{m} + a\vec{l}, \quad \vec{m}^* \rightarrow \vec{m}^* + a^*\vec{l}, \quad \vec{n} \rightarrow \vec{n} + a^*\vec{m} + a\vec{m}^* + aa^*\vec{l}; \\ \text{II: } & \vec{n} \rightarrow \vec{n}, \quad \vec{m} \rightarrow \vec{m} + b\vec{n}, \quad \vec{m}^* \rightarrow \vec{m}^* + b^*\vec{n}, \quad \vec{l} \rightarrow \vec{l} + b^*\vec{m} + b\vec{m}^* + bb^*\vec{n}; \\ \text{III: } & \vec{l} \rightarrow A\vec{l}, \quad \vec{n} \rightarrow A^{-1}\vec{n}, \quad \vec{m} \rightarrow e^{i\theta}\vec{m}, \quad \vec{m}^* \rightarrow e^{-i\theta}\vec{m}^*; \end{aligned}$$

where a and b are complex functions and A and θ are real functions.

For a rotation of class I, the spin coefficients (A.5) transform as

$$\begin{aligned} \kappa &\rightarrow \kappa, \quad \sigma \rightarrow \sigma + a\kappa, \quad \rho \rightarrow \rho + a^*\kappa, \quad \varepsilon \rightarrow \varepsilon + a^*\kappa, \\ \tau &\rightarrow \tau + a\rho + a^*\sigma + aa^*\kappa, \quad \pi \rightarrow \pi + 2a^*\varepsilon + (a^*)^2\kappa + Da^*, \\ \alpha &\rightarrow \alpha + a^*(\rho + \varepsilon) + (a^*)^2\kappa, \quad \beta \rightarrow \beta + a\varepsilon + a^*\sigma + aa^*\kappa, \\ \gamma &\rightarrow \gamma + a\alpha + a^*(\beta + \tau) + aa^*(\rho + \varepsilon) + (a^*)^2\sigma + a(a^*)^2\kappa, \\ \lambda &\rightarrow \lambda + a^*(2\alpha + \pi) + (a^*)^2(\rho + 2\varepsilon) + (a^*)^3\kappa + \delta^*a^* + a^*Da^*, \\ \mu &\rightarrow \mu + a\pi + 2a^*\beta + 2aa^*\varepsilon + (a^*)^2\sigma + a(a^*)^2\kappa + \delta a^* + aDa^*, \\ \nu &\rightarrow \nu + a\lambda + a^*(\mu + 2\gamma) + (a^*)^2(\tau + 2\beta) + (a^*)^3\sigma + aa^*(\pi + 2\alpha) \\ &\quad + a(a^*)^2(\rho + 2\varepsilon) + a(a^*)^3\kappa + (\Delta + a^*\delta + a\delta^* + aa^*D)a^*, \end{aligned} \tag{B.4}$$

and the Weyl (A.12) and Maxwell (A.17) scalars become

$$\begin{aligned} \Psi_0 &\rightarrow \Psi_0, \quad \Psi_1 \rightarrow \Psi_1 + a^*\Psi_0, \quad \Psi_2 \rightarrow \Psi_2 + 2a^*\Psi_1 + (a^*)^2\Psi_0, \\ \Psi_3 &\rightarrow \Psi_3 + 3a^*\Psi_2 + 3(a^*)^2\Psi_1 + (a^*)^3\Psi_0, \\ \Psi_4 &\rightarrow \Psi_4 + 4a^*\Psi_3 + 6(a^*)^2\Psi_2 + 4(a^*)^3\Psi_1 + (a^*)^4\Psi_0, \\ \phi_0 &\rightarrow \phi_0, \quad \phi_1 \rightarrow \phi_1 + a^*\phi_0, \quad \phi_2 \rightarrow \phi_2 + 2a^*\phi_1 + (a^*)^2\phi_0. \end{aligned} \tag{B.5}$$

Considering that $\rho, \mu, \gamma, \Psi_2$ and ϕ_1 are the only non-vanishing NP quantities for the black-string background in a Kinnersley null frame, the effect of an infinitesimal rotation by a parameter $a^{(1)}$ on the spin coefficients κ, σ, λ and ν is

$$\kappa^{(1)} \rightarrow \kappa^{(1)}, \quad \sigma^{(1)} \rightarrow \sigma^{(1)}, \quad \lambda^{(1)} \rightarrow \lambda^{(1)} + \delta^*a^{(1)*}, \quad \nu^{(1)} \rightarrow \nu^{(1)} + a^{(1)*}(\mu + 2\gamma) + \Delta a^{(1)*}. \tag{B.6}$$

and the effect on the first-order perturbations of the Weyl and Maxwell scalars is

$$\begin{aligned}\Psi_0^{(1)} &\rightarrow \Psi_0^{(1)}, & \Psi_1^{(1)} &\rightarrow \Psi_1^{(1)}, & \Psi_2^{(1)} &\rightarrow \Psi_2^{(1)}, & \Psi_3^{(1)} &\rightarrow \Psi_3^{(1)} + 3a^{(1)*}\Psi_2, \\ \Psi_4^{(1)} &\rightarrow \Psi_4^{(1)}, & \phi_0^{(1)} &\rightarrow \phi_0^{(1)}, & \phi_1^{(1)} &\rightarrow \phi_1^{(1)}, & \phi_2^{(1)} &\rightarrow \phi_2^{(1)} + 2a^{(1)*}\phi_1.\end{aligned}\tag{B.7}$$

The equations of transformation for a rotation of class II can be directly obtained from the formulas (B.4) and (B.5), simply by replacing a by b and considering that the interchange of \vec{l} and \vec{n} results in the transformation

$$\begin{aligned}\Psi_0 &\rightleftharpoons \Psi_4^*, & \Psi_1 &\rightleftharpoons \Psi_3^*, & \Psi_2 &\rightleftharpoons \Psi_2^*, & \phi_0 &\rightleftharpoons -\phi_2^*, & \phi_1 &\rightleftharpoons -\phi_1^*, \\ \kappa &\rightleftharpoons -\nu^*, & \sigma &\rightleftharpoons -\lambda^*, & \rho &\rightleftharpoons -\mu^*, & \tau &\rightleftharpoons -\pi^*, & \varepsilon &\rightleftharpoons -\gamma^*, & \alpha &\rightleftharpoons -\beta^*.\end{aligned}\tag{B.8}$$

In particular, for the perturbations of a charged black string in a Kinnersley null frame, an infinitesimal rotation of class II by a parameter $b^{(1)}$ leads to the changes

$$\begin{aligned}\kappa^{(1)} &\rightarrow \kappa^{(1)} + b^{(1)}\rho - Db^{(1)}, & \sigma^{(1)} &\rightarrow \sigma^{(1)} - \delta b^{(1)}, & \lambda^{(1)} &\rightarrow \lambda^{(1)}, & \nu^{(1)} &\rightarrow \nu^{(1)}, \\ \Psi_0^{(1)} &\rightarrow \Psi_0^{(1)}, & \Psi_1^{(1)} &\rightarrow \Psi_1^{(1)} + 3b^{(1)}\Psi_2, & \Psi_2^{(1)} &\rightarrow \Psi_2^{(1)}, & \Psi_3^{(1)} &\rightarrow \Psi_3^{(1)}, \\ \Psi_4^{(1)} &\rightarrow \Psi_4^{(1)}, & \phi_0^{(1)} &\rightarrow \phi_0^{(1)} + 2b^{(1)}\phi_1, & \phi_1^{(1)} &\rightarrow \phi_1^{(1)}, & \phi_2^{(1)} &\rightarrow \phi_2^{(1)}.\end{aligned}\tag{B.9}$$

The effect of a rotation of class III on the Newman-Penrose quantities is the following:

$$\begin{aligned}\Psi_0 &\rightarrow A^2 e^{2i\theta} \Psi_0; & \Psi_1 &\rightarrow A e^{i\theta} \Psi_1; & \Psi_2 &\rightarrow \Psi_2; \\ \Psi_3 &\rightarrow A^{-1} e^{-i\theta} \Psi_3; & \Psi_4 &\rightarrow A^{-2} e^{-2i\theta} \Psi_4; \\ \phi_0 &\rightarrow A e^{i\theta} \phi_0; & \phi_1 &\rightarrow \phi_1; & \phi_2 &\rightarrow A^{-1} e^{-i\theta} \phi_2; \\ \kappa &\rightarrow A^2 e^{i\theta} \kappa; & \sigma &\rightarrow A e^{2i\theta} \sigma; & \rho &\rightarrow A \rho; \\ \pi &\rightarrow e^{-i\theta} \pi; & \lambda &\rightarrow A^{-1} e^{-2i\theta} \lambda; & \mu &\rightarrow A^{-1} \mu; \\ \gamma &\rightarrow A^{-1} \gamma + \frac{1}{2} A^{-2} \Delta A + \frac{1}{2} i A^{-1} \Delta \theta; & \nu &\rightarrow A^{-2} e^{-i\theta} \nu; \\ \varepsilon &\rightarrow A \varepsilon + \frac{1}{2} D A + \frac{1}{2} i A D \theta; & \tau &\rightarrow e^{i\theta} \tau; \\ \alpha &\rightarrow e^{-i\theta} \alpha + \frac{1}{2} A^{-1} e^{-i\theta} \delta^* A + \frac{1}{2} i e^{-i\theta} \delta^* \theta; \\ \beta &\rightarrow e^{-i\theta} \beta + \frac{1}{2} A^{-1} e^{i\theta} \delta A + \frac{1}{2} i e^{i\theta} \delta \theta.\end{aligned}\tag{B.10}$$

Also in relation to a rotation of class III, a scalar T is said to have a conformal weight C and a spin weight S if it transforms as

$$T \rightarrow A^C e^{iS\theta} T.\tag{B.11}$$

For a rotating charged black string, an infinitesimal rotation of class III leaves the spin coefficients κ , σ , λ , ν and the Weyl and Maxwell scalars unchanged, since the transformed quantities are proportional to the original ones and the quantities are either vanishing in the background spacetime (the case of κ , σ , λ , ν , Ψ_0 , Ψ_1 , Ψ_3 , Ψ_4 , ϕ_0 and ϕ_2) or they are invariant under class-III rotations (the case of Ψ_2 and ϕ_1).

Appendix C. The transformation theory with source terms

In this appendix, we extend the Chandrasekhar transformation theory [60–62] to take into account the presence of source terms in the fundamental wave equations. The aim is to find transformations that relate the solutions of equations of the form

$$\Lambda^2 Y_{+i} + P_i \Lambda_+ Y_{+i} - Q_i Y_{+i} = \mathfrak{S}_{+i} \quad (\text{C.1})$$

to solutions of the one-dimensional Schrödinger-like equations

$$\Lambda^2 Z_i = V_i Z_i + \mathcal{F}_i, \quad (\text{C.2})$$

where

$$P_i = \frac{d}{dr_*} \ln \left(\frac{r^8}{\mathcal{D}_i} \right), \quad (\text{C.3})$$

and Q_i and V_i are, for the moment, unspecified functions of the radial coordinate, and the operators Λ^2 and Λ_{\pm} are defined in section 3.4.

The only difference between the form of equations (62) for Y_{+i} (with source terms \mathfrak{S}_{+i}) and the form of equations (69) for Y_{-i} (with source terms \mathfrak{S}_{-i}) is the presence of the operator Λ_+ in one case and the operator Λ_- in the other. Then, the equations of a transformation theory developed for Y_{+i} can be obtained from those for Y_{-i} by the exchange $+\varpi \rightarrow -\varpi$, and vice-versa. In order to simplify the notation, we restrict the study to equation (C.1) and write Y_i and \mathfrak{S}_i in place of Y_{+i} and \mathfrak{S}_{+i} .

As usual in the transformation theory, let's assume that Y_i is given by a combination of Z_i and its derivative of the form

$$Y_i = f_i \Lambda_- \Lambda_- Z_i + \Xi_i \Lambda_- Z_i, \quad (\text{C.4})$$

or, considering that $\Lambda_- = \Lambda_+ - 2i\varpi$ and substituting $\Lambda^2 Z_i$ from (C.2), we have

$$Y_i = f_i V_i Z_i + (\Xi_i - 2i\varpi f_i) \Lambda_- Z_i + f_i \mathcal{F}_i, \quad (\text{C.5})$$

where f_i and Ξ_i are functions of r (or r_*) to be determined.

Applying the operator Λ_+ to equation (C.5) and making use of equation (C.2), we find

$$\begin{aligned} \Lambda_+ Y_i &= \left[\frac{d}{dr_*} (f_i V_i) + \Xi_i V_i \right] Z_i + \left[f_i V_i + \frac{d}{dr_*} (\Xi_i - 2i\varpi f_i) \right] \Lambda_- Z_i + \mathcal{L}_{-i} \mathcal{F}_i \\ &= -\beta_i \frac{\mathcal{D}_i}{r^8} Z_i + R_i \Lambda_- Z_i + \mathcal{L}_{-i} \mathcal{F}_i, \end{aligned} \quad (\text{C.6})$$

where the operators \mathcal{L}_{+i} and \mathcal{L}_{-i} are defined by

$$\mathcal{L}_{\pm i} = \Xi_i + \frac{d f_i}{dr_*} + f_i \Lambda_{\pm}, \quad (\text{C.7})$$

and the functions β_i and R_i are given by

$$\beta_i = -\frac{r^8}{\mathcal{D}_i} \left[\frac{d}{dr_*} (f_i V_i) + \Xi_i V_i \right], \quad (\text{C.8})$$

$$R_i = f_i V_i + \frac{d}{dr_*} (\Xi_i - 2i\varpi f_i). \quad (\text{C.9})$$

A similar procedure, with Λ_+ now applied to equation (C.6) and the use of (C.2) to simplify the resulting expression, yields

$$\begin{aligned} \Lambda_+ \Lambda_+ Y_i = & -\beta_i \frac{\mathcal{D}_i}{r^8} (\Lambda_- + 2i\varpi) Z_i - \beta_i Z_i \frac{d}{dr_*} \left(\frac{\mathcal{D}_i}{r^8} \right) \\ & - \frac{d\beta_i}{dr_*} \frac{\mathcal{D}_i}{r^8} Z_i + R_i V_i Z_i + \frac{dR_i}{dr_*} \Lambda_- Z_i + (R_i + \Lambda_+ \mathcal{L}_{-i}) \mathcal{F}_i. \end{aligned} \quad (\text{C.10})$$

On the other hand, it follows from equation (C.1) that

$$\Lambda_+ \Lambda_+ Y_i = \Lambda^2 Y_i + 2i\varpi \Lambda_+ Y_i = -(P_i - 2i\varpi) \Lambda_+ Y_i + Q_i Y_i + \mathfrak{S}_i, \quad (\text{C.11})$$

or, substituting Y_i and $\Lambda_+ Y_i$ from equations (C.5) and (C.6), we find

$$\begin{aligned} \Lambda_+ \Lambda_+ Y_i = & -(P_i - 2i\varpi) \left[-\beta_i \frac{\mathcal{D}_i}{r^8} Z_i + R_i \Lambda_- Z_i \right] + Q_i [f_i V_i Z_i + (\Xi_i - 2i\varpi f_i) \Lambda_- Z_i] \\ & - (P_i - 2i\varpi) \mathcal{L}_{-i} \mathcal{F}_i + Q_i f_i \mathcal{F}_i + \mathfrak{S}_i. \end{aligned} \quad (\text{C.12})$$

Considering that equations (C.10) and (C.12) must be equal to each other, one can compare the terms which do not contain Z_i , as well as the coefficients of Z_i and $\Lambda_- Z_i$, in both equations. After some algebra, we obtain

$$-\frac{\mathcal{D}_i}{r^8} \frac{d\beta_i}{dr_*} = (Q_i f_i - R_i) V_i, \quad (\text{C.13})$$

$$\frac{d}{dr_*} \left(\frac{r^8}{\mathcal{D}_i} R_i \right) = \frac{r^8}{\mathcal{D}_i} [Q_i (\Xi_i - 2i\varpi f_i) + 2i\varpi R_i] + \beta_i, \quad (\text{C.14})$$

$$\mathfrak{S}_i = (P_i + \Lambda_-) \mathcal{L}_{-i} \mathcal{F}_i - (Q_i f_i - R_i) \mathcal{F}_i. \quad (\text{C.15})$$

As shown by Chandrasekhar in [60–62], the system of equations (C.8), (C.9), (C.13) and (C.14) admits the integral

$$\frac{r^8}{\mathcal{D}_i} R_i f_i V_i + \beta_i (\Xi_i - 2i\varpi f_i) = K_i = \text{constant}. \quad (\text{C.16})$$

This equation, in turn, allows to write the inverse of relations (C.5) and (C.6) as

$$K_i Z_i = \frac{r^8}{\mathcal{D}_i} R_i Y_i - \frac{r^8}{\mathcal{D}_i} (\Xi_i - 2i\varpi f_i) \Lambda_+ Y_i - \frac{r^8}{\mathcal{D}_i} [R_i f_i - (\Xi_i - 2i\varpi f_i) \mathcal{L}_{-i}] \mathcal{F}_i, \quad (\text{C.17})$$

$$K_i \Lambda_- Z_i = \beta_i Y_i + \frac{r^8}{\mathcal{D}_i} f_i V_i \Lambda_+ Y_i - \beta_i f_i \mathcal{F}_i - \frac{r^8}{\mathcal{D}_i} f_i V_i \mathcal{L}_{-i} \mathcal{F}_i. \quad (\text{C.18})$$

In the study of perturbations of Schwarzschild and Reissner-Nordström black holes [61, 62] (without sources), it is found that equations (C.8), (C.9), (C.13), (C.14) and (C.16) are satisfied by a special set of transformations for which

$$\beta_i = \text{constant} \quad \text{and} \quad f_i = 1. \quad (\text{C.19})$$

Since these equations do not change with the inclusion of \mathfrak{S}_i and \mathcal{F}_i in the wave equations (C.1) and (C.2), the same assumptions can be considered here and will lead to consistent solutions. For a given function Q_i , there are conditions to be satisfied by R_i , V_i , Ξ_i , β_i and K_i for the existence of transformations compatible with (C.19). Such conditions are presented below.

Imposing $\beta_i = \text{constant}$ and $f_i = 1$ on (C.13), we find that $R_i = Q_i$. In conjunction with (C.14) and (C.15), these conditions yield

$$\frac{d}{dr_*} \left(\frac{r^8}{\mathcal{D}_i} Q_i \right) = \frac{r^8}{\mathcal{D}_i} Q_i \Xi_i + \beta_i \quad (\text{C.20})$$

and

$$\mathfrak{S}_i = (P_i + \Lambda_-)(\Xi_i + \Lambda_-)\mathcal{F}_i = (P_i + \Lambda_-)\mathcal{L}_{-i}\mathcal{F}_i. \quad (\text{C.21})$$

The basic set of equations are completed by (C.9) and (C.16), which can be put into the form

$$V_i = Q_i - \frac{d}{dr_*} \Xi_i, \quad (\text{C.22})$$

$$\left(\frac{r^8}{\mathcal{D}_i} Q_i \right) V_i + \beta_i \Xi_i = K_i + 2i\varpi \beta_i = \text{constant} \equiv \varkappa_i. \quad (\text{C.23})$$

Introducing the function $F_i = (r^8/\mathcal{D}_i)Q_i$ and using the expression (C.22) for V_i , we can rewrite equations (C.20) and (C.23) as

$$\Xi_i = \frac{1}{F_i} \left(\frac{dF_i}{dr_*} - \beta_i \right) \quad \text{and} \quad F_i \left(Q_i - \frac{d\Xi_i}{dr_*} \right) + \beta_i \Xi_i = \varkappa_i. \quad (\text{C.24})$$

Therefore, eliminating Ξ_i from the above equations and performing some simplifications, we obtain

$$\frac{1}{F_i} \left(\frac{dF_i}{dr_*} \right)^2 - \frac{d^2 F_i}{dr_*^2} + \frac{\mathcal{D}_i}{r^8} F_i^2 = \frac{\beta_i^2}{F_i} + \varkappa_i. \quad (\text{C.25})$$

The last equation shows that a necessary and sufficient condition for the compatibility of the transformation equations with (C.19) is the existence of constants β_i and \varkappa_i such that equation (C.25) is satisfied by the given function $Q_i = (\mathcal{D}_i/r^8)F_i$. Considering that β_i appears in equation (C.25) as β_i^2 , we have a pair of the so-called dual transformations, one of them generated by $\beta_i^{(+)} = +\beta_i$ and the other one by $\beta_i^{(-)} = -\beta_i$.

Using the superscripts (\pm) to distinguish between the transformations with $+\beta_i$ and $-\beta_i$, we have

$$\Xi_i^{(\pm)} = \frac{1}{F_i} \left(\frac{dF_i}{dr_*} \mp \beta_i \right) \quad (\text{C.26})$$

and

$$V_i^{(\pm)} = Q_i - \frac{d\Xi_i^{(\pm)}}{dr_*}. \quad (\text{C.27})$$

Substituting $\Xi_i^{(\pm)}$ from equation (C.26) in (C.27) and making use of the second of equations (C.24), we find

$$V_i^{(\pm)} = Q_i - \frac{d}{dr_*} \left[\frac{1}{F_i} \left(\frac{dF_i}{dr_*} \mp \beta_i \right) \right] = \frac{\varkappa_i}{F_i} \mp \frac{\beta_i}{F_i^2} \left(\frac{dF_i}{dr_*} \mp \beta_i \right). \quad (\text{C.28})$$

So, introducing the function $\mathfrak{f}_i = 1/F_i$, we obtain the following formulas for the potentials:

$$V_i^{(\pm)} = \pm \beta_i \frac{d\mathfrak{f}_i}{dr_*} + \beta_i^2 \mathfrak{f}_i^2 + \varkappa_i \mathfrak{f}_i. \quad (\text{C.29})$$

The associated transformations relating Y_i to $Z_i^{(\pm)}$ are given, in explicit form, by

$$\begin{aligned} Y_i &= V_i^{(\pm)} Z_i^{(\pm)} + (\Xi_i^{(\pm)} - 2i\varpi) \Lambda_- Z_i^{(\pm)} + \mathcal{F}_i^{(\pm)}, \\ \Lambda_+ Y_i &= \mp \beta_i \frac{\mathcal{D}_i}{r^8} Z_i^{(\pm)} + Q_i \Lambda_- Z_i^{(\pm)} + \mathcal{L}_{-i}^{(\pm)} \mathcal{F}_i^{(\pm)}, \end{aligned} \quad (\text{C.30})$$

and

$$\begin{aligned} K_i^{(\mp)} Z_i^{(\pm)} &= \frac{r^8}{\mathcal{D}_i} Q_i Y_i - \frac{r^8}{\mathcal{D}_i} (\Xi_i^{(\pm)} - 2i\varpi) \Lambda_+ Y_i - \frac{r^8}{\mathcal{D}_i} [Q_i - (\Xi_i^{(\pm)} - 2i\varpi) \mathcal{L}_{-i}^{(\pm)}] \mathcal{F}_i^{(\pm)}, \\ K_i^{(\mp)} \Lambda_- Z_i^{(\pm)} &= \pm \beta_i Y_i + \frac{r^8}{\mathcal{D}_i} V_i^{(\pm)} \Lambda_+ Y_i \mp \beta_i \mathcal{F}_i^{(\pm)} - \frac{r^8}{\mathcal{D}_i} V_i^{(\pm)} \mathcal{L}_{-i}^{(\pm)} \mathcal{F}_i^{(\pm)}, \end{aligned} \quad (\text{C.31})$$

where $K_i^{(\pm)} = \varkappa_i \pm 2i\varpi \beta_i$, and the operators $\mathcal{L}_{-i}^{(\pm)}$ are given by

$$\mathcal{L}_{-i}^{(\pm)} = \Xi_i^{(\pm)} + \Lambda_-. \quad (\text{C.32})$$

The source terms appearing in equations (C.1) and (C.2) are related by

$$\mathfrak{S}_i = (P_i + \Lambda_-) \mathcal{L}_{-i}^{(\pm)} \mathcal{F}_i^{(\pm)}. \quad (\text{C.33})$$

The present form of such equations is not appropriate for a transformation theory, since they give us a prescription to compute the known terms \mathfrak{S}_i from the unknown source terms $\mathcal{F}_i^{(\pm)}$, while the opposite would be expected. In order to invert these equations, we follow Sasaki and Nakamura [95] and look for a set of functions $h_i^{(\pm)}(r)$ and $g_i^{(\pm)}(r)$ such that

$$\mathfrak{S}_i = \frac{1}{h_i^{(\pm)} g_i^{(\pm)}} \Lambda_- h_i^{(\pm)} \Lambda_- (g_i^{(\pm)} \mathcal{F}_i^{(\pm)}). \quad (\text{C.34})$$

Comparing (C.34) to (C.33), we find the functions $h_i^{(\pm)}(r)$ and $g_i^{(\pm)}(r)$ satisfy the following set of inhomogeneous coupled equations:

$$2 \frac{d}{dr_*} \ln g_i^{(\pm)} + \frac{d}{dr_*} \ln h_i^{(\pm)} = P_i + \Xi_i^{(\pm)}; \quad (\text{C.35})$$

$$\frac{d^2}{dr_*^2} \ln g_i^{(\pm)} + \left(\frac{d}{dr_*} \ln g_i^{(\pm)} + \frac{d}{dr_*} \ln h_i^{(\pm)} \right) \frac{d}{dr_*} \ln g_i^{(\pm)} = P_i \Xi_i^{(\pm)} + \frac{d}{dr_*} \Xi_i^{(\pm)}. \quad (\text{C.36})$$

This system of equations can be analytically solved and the solutions are given by

$$h_i^{(\pm)}(r) = \frac{\mathcal{V}_i^{(\pm)}}{Q_i} \left(C_i^{(\pm)} - \int^{r_*} \frac{Q_i(r'_*)}{\mathcal{V}_i^{(\pm)}(r'_*)} dr'_* \right)^2, \quad (\text{C.37})$$

$$g_i^{(\pm)}(r) = \frac{F_i}{\mathcal{V}_i^{(\pm)}} \left(C_i^{(\pm)} - \int^{r_*} \frac{Q_i(r'_*)}{\mathcal{V}_i^{(\pm)}(r'_*)} dr'_* \right)^{-1}, \quad (\text{C.38})$$

where the functions $\mathcal{V}_i^{(\pm)}$ are defined as

$$\mathcal{V}_i^{(\pm)} = \exp \left(\pm \beta_i \int^{r_*} f_i(r'_*) dr'_* \right) \quad (\text{C.39})$$

and $C_i^{(\pm)}$ are integration constants, which can be chosen as zero by convenience.

Equation (C.34) can be further simplified with the introduction of the new functions

$$\mathcal{W}_i^{(\pm)} = g_i^{(\pm)} \mathcal{F}_i^{(\pm)} e^{-i\varpi r_*}. \quad (\text{C.40})$$

In terms of $\mathcal{W}_i^{(\pm)}$, equation (C.34) becomes

$$\frac{d}{dr_*} \left[h_i^{(\pm)} \frac{d}{dr_*} \mathcal{W}_i^{(\pm)} \right] = h_i^{(\pm)} g_i^{(\pm)} \mathfrak{S}_i e^{-i\varpi r_*}, \quad (\text{C.41})$$

so that the problem of expressing $\mathcal{F}_i^{(\pm)}$ in terms of \mathfrak{S}_i is reduced to quadratures.

To complete this analysis, we notice that, as in the standard transformation theory, the solutions for $Z_i^{(+)}$ and $Z_i^{(-)}$ can be related to each other. In fact, taking from (C.30) the expressions for Y_i and $\Lambda_+ Y_i$ in terms of $Z_i^{(+)}$, $\Lambda_- Z_i^{(+)}$, and $\mathcal{F}_i^{(+)}$ and substituting them into the first of equations (C.31) for $Z_i^{(-)}$, we find

$$\begin{aligned} K_i^{(+)} Z_i^{(-)} &= [F_i V_i^{(+)} + \beta_i (\Xi_i^{(+)} - 2i\varpi) - \beta_i (\Xi_i^{(+)} - \Xi_i^{(-)})] Z_i^{(+)} + F_i (\Xi_i^{(+)} - \Xi_i^{(-)}) \Lambda_- Z_i^{(+)} \\ &\quad + \frac{r^8}{\mathcal{D}_i} (\Xi_i^{(-)} - 2i\varpi) (\mathcal{L}_{-i}^{(-)} \mathcal{F}_i^{(-)} - \mathcal{L}_{-i}^{(+)} \mathcal{F}_i^{(+)}) + F_i (\mathcal{F}_i^{(+)} - \mathcal{F}_i^{(-)}). \end{aligned} \quad (\text{C.42})$$

However, since

$$\mathfrak{S}_i = (P_i + \Lambda_-) \mathcal{L}_{-i}^{(-)} \mathcal{F}_i^{(-)} = (P_i + \Lambda_-) \mathcal{L}_{-i}^{(+)} \mathcal{F}_i^{(+)}, \quad (\text{C.43})$$

we have

$$\mathcal{L}_{-i}^{(-)} \mathcal{F}_i^{(-)} - \mathcal{L}_{-i}^{(+)} \mathcal{F}_i^{(+)} = \mathcal{A}_i \frac{\mathcal{D}_i}{r^8} e^{i\varpi r_*}, \quad (\text{C.44})$$

where \mathcal{A}_i are non-vanishing arbitrary integration constants. From equations (C.23) and (C.26), it also follows that

$$F_i V_i^{(+)} + \beta_i (\Xi_i^{(+)} - 2i\varpi) = K_i^{(+)} \quad \text{and} \quad \Xi_i^{(+)} - \Xi_i^{(-)} = -\frac{2\beta_i}{F_i}. \quad (\text{C.45})$$

Substituting these relations into (C.42), we obtain the desired expression

$$K_i^{(+)} Z_i^{(-)} = \left(\varkappa_i + 2 \frac{\beta_i^2}{F_i} \right) Z_i^{(+)} - 2 \beta_i \frac{dZ_i^{(+)}}{dr_*} + \mathcal{A}_i (\Xi_i^{(-)} - 2i\varpi) e^{i\varpi r_*} + F_i (\mathcal{F}_i^{(+)} - \mathcal{F}_i^{(-)}). \quad (\text{C.46})$$

The inverse of this relation is given by

$$K_i^{(-)} Z_i^{(+)} = \left(\varkappa_i + 2 \frac{\beta_i^2}{F_i} \right) Z_i^{(-)} + 2 \beta_i \frac{dZ_i^{(-)}}{dr_*} - \mathcal{A}_i \left(\Xi_i^{(+)} - 2i\varpi \right) e^{i\varpi r_*} + F_i \left(\mathcal{F}_i^{(-)} - \mathcal{F}_i^{(+)} \right). \quad (\text{C.47})$$

This completes an important result generalizing the Chandrasekhar transformation theory including source terms, useful for applications to first order perturbations of nonempty spacetimes.

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